Discrete Algebraic Structures

WiSe 2025/2026

Prof. Dr. Antoine Wiehe
Research Group for Theoretical Computer Science

Abstract	Concrete
Structures with multiplication, neutral elements	(relations, $\circ$, Id) $(\{0,1\}^*, \cdot, \epsilon)$ $\boxed{\text{Monoids}}$
Structures with multiplication, inverses, neutral elements	Real Numbers: $\times, 1/x, 1$ Matrices: $\times, M^{-1}, I$ Bijective Functions: $\circ, f^{-1}, \mathrm{Id}_A$ Modular arithmetic: $\times, [a]_d^{-1}, [1]_d$ $\boxed{\text{Groups}}$
Structures with addition and multiplication	$(\mathbb{R}, +, \times)$ $(\mathbb{Z}/d\mathbb{Z}, +, \times)$ $(\mathbb{R}^{n\times n}, +, \times)$ $(\mathbb{R}[X], +, \times)$ $\boxed{\text{Rings}}$
Structures with addition and scalar multiplication	$(\mathbb{R}^n, +, \lambda\cdot)$ $\boxed{\text{Vector spaces}}$

Definition of abstract object that generalizes something we know ⟵ Concrete examples we want to study

↓

for example, Fermat's little theorem

Identify some properties that are important in the concrete case ⟶ General theorems about all our examples

Abstract	**Concrete**	
Structures with multiplication, neutral elements	(relations, $\circ$, Id) ($\{0,1\}^*, \cdot, \epsilon$)	Monoids
Structures with multiplication, inverses, neutral elements	Real Numbers: $\times, 1/x, 1$ Matrices: $\times, M^{-1}, I$ Bijective Functions: $\circ, f^{-1}, \mathrm{Id}_A$ Modular arithmetic: $\times, [a]_d^{-1}, [1]_d$	Groups
Structures with addition and multiplication	$(\mathbb{R}, +, \times)$ $(\mathbb{Z}/d\mathbb{Z}, +, \times)$ $(\mathbb{R}^{n\times n}, +, \times)$ $(\mathbb{R}[X], +, \times)$	Rings
Structures with addition and scalar multiplication	$(\mathbb{R}^n, +, \lambda\cdot)$	Vector spaces

Plan for today:

- a bit more about groups: homomorphisms, how to compare groups
- structures with several binary operations: rings
- polynomials

Goal:

- overview of algebra for CS
- understand necessary notions for error-correcting codes (for next week)

Definition. An internal composition law or binary operation on A is a function $\circ\colon A^2 \to A$.
We write $a \circ b$ instead of $\circ(a, b)$.
$(A, \circ)$ is a **monoid** if $\circ$ is associative and has a neutral element
$(A, \circ)$ is a **group** if $\circ$ is associative, has a neutral element, and every $a \in A$ has an inverse

Example.

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

This is:

- A monoid?
- A group?
- Neither a monoid nor a group?

Definition. An internal composition law or binary operation on A is a function $\circ\colon A^2 \to A$.
We write $a \circ b$ instead of $\circ(a, b)$.
$(A, \circ)$ is a **monoid** if $\circ$ is associative and has a neutral element
$(A, \circ)$ is a **group** if $\circ$ is associative, has a neutral element, and every $a \in A$ has an inverse

Example.

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$a \mapsto 0$
$b \mapsto 1$
$c \mapsto 2$

Example.

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Two different groups! But.. are they really different?

Definition. An internal composition law or binary operation on A is a function $\circ\colon A^2 \to A$. We write $a \circ b$ instead of $\circ(a, b)$.
$(A, \circ)$ is a **monoid** if $\circ$ is associative and has a neutral element
$(A, \circ)$ is a **group** if $\circ$ is associative, has a neutral element, and every $a \in A$ has an inverse

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024} \qquad 2^{1/2} \times 2^{1/4} \times \cdots \times 2^{1/1024}$$

$$(\mathbb{R}, +) \xrightarrow{x \mapsto 2^x} (\mathbb{R}_{>0}, \times)$$

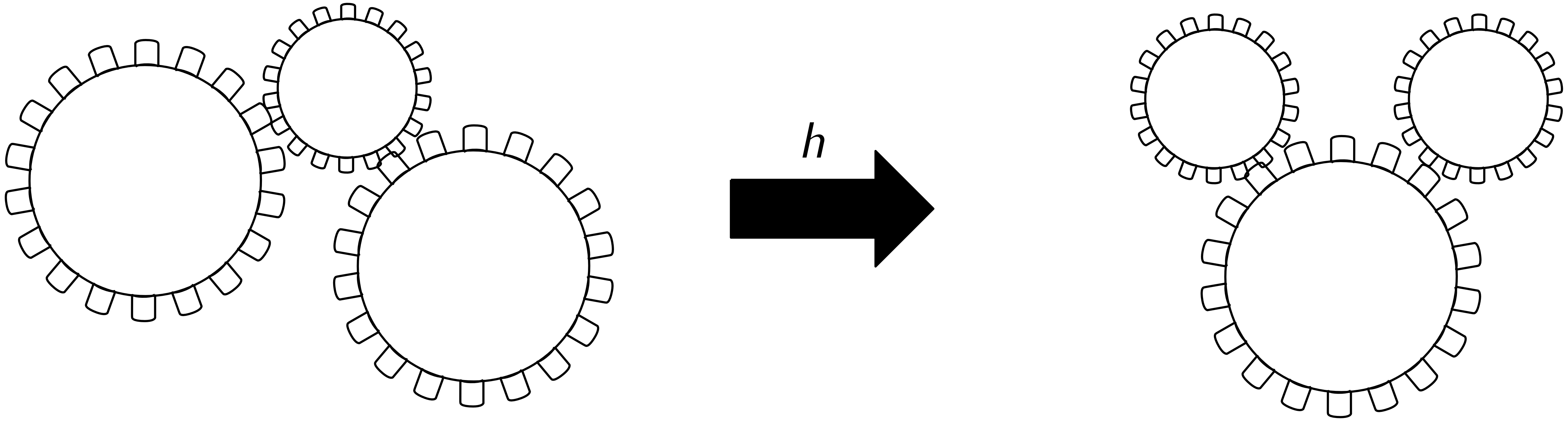
Two different groups! But.. are they really different?

Definition. An internal composition law or binary operation on A is a function $\circ\colon A^2 \to A$. We write $a \circ b$ instead of $\circ(a, b)$.
$(A, \circ)$ is a **monoid** if $\circ$ is associative and has a neutral element
$(A, \circ)$ is a **group** if $\circ$ is associative, has a neutral element, and every $a \in A$ has an inverse

$$1 + 2 + \cdots + 7 \qquad [1]_8 + [2]_8 + \cdots + [7]_8$$

$$(\mathbb{Z}, +) \xrightarrow[x \mapsto [x]_8]{} (\mathbb{Z}/8\mathbb{Z}, +)$$

Definition. An internal composition law or binary operation on A is a function $\circ\colon A^2 \to A$.
We write $a \circ b$ instead of $\circ(a, b)$.
$(A, \circ)$ is a **monoid** if $\circ$ is associative and has a neutral element
$(A, \circ)$ is a **group** if $\circ$ is associative, has a neutral element, and every $a \in A$ has an inverse



Computation in $(A, \circ)$

$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024}$

$1 + 2 + \cdots + 7$

Computation in $(B, \Box)$

$2^{1/2} \times 2^{1/4} \times \cdots \times 2^{1/1024}$

$[1]_8 + [2]_8 + \cdots + [7]_8$

Definition. Let $(A, \circ)$ and $(B, \Box)$ be two monoids.
A homomorphism $h\colon (A, \circ) \to (B, \Box)$ is a function such that
$$h(a \circ a') = h(a) \Box h(a')$$

From Math 1:

Definition. A map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is called linear if for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have:
$$f(x+y) = f(x)+f(y)$$
$$f(\lambda x) = \lambda f(x)$$

$\rightsquigarrow$ a linear map is a homomorphism $(\mathbb{R}^n, +) \to (\mathbb{R}^m, +)$!

Definition. Let $(A, \circ)$ and $(B, \square)$ be two monoids.
A homomorphism $h\colon (A, \circ) \to (B, \square)$ is a function such that

$$h(a \circ a') = h(a) \square h(a')$$

Examples.

- Linear maps $\mathbb{R}^n \to \mathbb{R}^m$
- Exponential map: $x \mapsto e^x$ is a homomorphism $(\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$
- Logarithm?
- The "mod d" function $x \mapsto [x]_d$ is a homomorphism $(\mathbb{Z}, +) \to (\mathbb{Z}/d\mathbb{Z}, +)$

For $x \in \{0, \dots, d-1\}$, define $h([x]_d) = x$.
This is a function $\mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}$.
Is this a homomorphism $(\mathbb{Z}/d\mathbb{Z}, +) \to (\mathbb{Z}, +)$?

Homomorphisms between structures

Definition. Let $(A, \circ)$ and $(B, \square)$ be two monoids.
A homomorphism $h\colon (A, \circ) \to (B, \square)$ is a function such that

$$h(a \circ a') = h(a) \square h(a')$$

Examples.

- Linear maps $\mathbb{R}^n \to \mathbb{R}^m$
- Exponential map: $x \mapsto e^x$ is a homomorphism $(\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$
- Logarithm?
- The "mod d" function $x \mapsto [x]_d$ is a homomorphism $(\mathbb{Z}, +) \to (\mathbb{Z}/d\mathbb{Z}, +)$

Consider det as a function from $n \times n$-matrices to $\mathbb{R}$.
Is it a homomorphism:

- $(\mathbb{R}^{n\times n}, +) \to (\mathbb{R}, +)$?
- $(\mathbb{R}^{n\times n}, \times) \to (\mathbb{R}, +)$?
- $(\mathbb{R}^{n\times n}, \times) \to (\mathbb{R}, \times)$?
- $(\mathbb{R}^{n\times n}, \times) \to (\mathbb{R}, +)$?

Definition. Let $(A, \circ)$ and $(B, \square)$ be two monoids.
A homomorphism $h\colon (A, \circ) \to (B, \square)$ is a function such that

$$h(a \circ a') = h(a) \square h(a')$$

Examples.

- Linear maps $\mathbb{R}^n \to \mathbb{R}^m$
- Exponential map: $x \mapsto e^x$ is a homomorphism $(\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$
- Logarithm?
- The "mod d" function $x \mapsto [x]_d$ is a homomorphism $(\mathbb{Z}, +) \to (\mathbb{Z}/d\mathbb{Z}, +)$

If h is a bijection, we say that it is an isomorphism.

Example. $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ is a group (with matrix multiplication).
Which group has an isomorphism with G?

- $(\mathbb{Z}/2\mathbb{Z}, +)$
- $(\mathbb{Z}, +)$
- $(\mathbb{Z}/3\mathbb{Z}, +)$

Definition. Let $(A, \circ)$ and $(B, \square)$ be two monoids.
A homomorphism $h\colon (A, \circ) \to (B, \square)$ is a function such that
$$h(a \circ a') = h(a) \square h(a')$$

Examples.

- Linear maps $\mathbb{R}^n \to \mathbb{R}^m$
- Exponential map: $x \mapsto e^x$ is a homomorphism $(\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$
- Logarithm?
- The "mod d" function $x \mapsto [x]_d$ is a homomorphism $(\mathbb{Z}, +) \to (\mathbb{Z}/d\mathbb{Z}, +)$

If h is a bijection, we say that it is an isomorphism.
Up to isomorphism, there is only one group of size 2.

n	1	2	3	4	5	6	7	8	9	10	11	12
number of groups	1	1	1	2	1	2	11	5	2	2	1	5

Message:

- if the "thing" you are studying is a group, then there is a lot of structure to exploit
- even if your group is not we saw in class, it could be isomorphic to one

Rings, Fields, and Polynomials

Abstract	Concrete	
Structures with multiplication, neutral elements	Relations: $\circ$, Id $(\{0,1\}^*, \cdot, \epsilon)$	Monoids
Structures with multiplication, inverses, neutral elements	Real Numbers: $\times, 1/x, 1$ Matrices: $\times, M^{-1}, I$ Bijective Functions: $\circ, f^{-1}, \mathrm{Id}_A$ Modular arithmetic: $\times, [a]_d^{-1}, [1]_d$	Groups
Structures with addition and multiplication	$(\mathbb{R}, +, \times)$ $(\mathbb{Z}/d\mathbb{Z}, +, \times)$ $(\mathbb{R}^{n\times n}, +, \times)$ $(\mathbb{R}[X], +, \times)$	Rings
Structures with addition and scalar multiplication	$(\mathbb{R}^n, +, \lambda\cdot)$	Vector spaces

What can we say about how $+$ and $\times$ relate in our examples?

Definition. Let $+$ and $\times$ be internal composition laws on A.
We say that $\times$ distributes over $+$ if for all $a, b, c \in A$

$$a \times (b + c) = a \times b + a \times c \qquad (b + c) \times a = b \times a + c \times a$$

Definition. Let $+$ and $\times$ be internal composition laws on A.
We say that $\times$ distributes over $+$ if for all $a, b, c \in A$

$$a \times (b + c) = a \times b + a \times c \qquad (b + c) \times a = b \times a + c \times a$$

- All addition/multiplication operations you know
- $\varphi \wedge (\psi \vee \theta) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \theta)$
- $\varphi \vee (\psi \wedge \theta) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \theta)$
- same with $\cap$ and $\cup$

Definition. Let $+$ and $\times$ be binary operations on A.
Then $(A, +, \times)$ is a **ring** if:
- $(A, +)$ is a commutative group,
- $(A, \times)$ is monoid,
- $\times$ distributes over $+$.

Exercise: for each of the following, think about why it is a ring.
- $(\{0, 1\}, \wedge, \vee)$
- $(\{0, 1\}, \vee, \wedge)$
- $(\mathbb{Z}/d\mathbb{Z}, +, \times)$
- $(\mathbb{R}^{n\times n}, +, \times)$

Neutral element for $+$: written 0
Neutral element for $\times$: written 1

$\boxed{\text{Groups}} \subseteq \boxed{\text{Monoids}}$ $\boxed{\text{Fields}} \subseteq \boxed{\text{Rings}}$

Group "=" monoid with subtraction Field "=" commutative ring with division

Definition. Let $(R, +, \times)$ be a ring. We call it a **field** if:

- every $a \neq 0$ has a multiplicative inverse: some a^{-1} such that $a \times a^{-1} = 1$
- $\times$ is commutative

Examples. The following are fields:

- $(\mathbb{R}, +, \times)$
- $(\mathbb{Q}, +, \times)$
- Rational fractions (with addition and multiplication)
- $(\mathbb{Z}/d\mathbb{Z}, +, \times)$?

Most of what you learn for $\mathbb{R}^n$ is true for every $\mathbb{K}^n$ if $\mathbb{K}$ is a field
Since $\mathbb{Z}/2\mathbb{Z}$ is a field so:

- can talk about linear maps and inverses and ... with vectors in $\{0, 1\}^n$
- can talk about Fourier transforms
- a lot of modern CS (pratice and theory) does not exist without this algebraic concept

Polynomials

Definition. Let $(R, +, \times)$ be a ring. A polynomial with coefficients in R is an expression of the form
$$a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m$$
where $a_0, \ldots, a_m \in R$.

- a_i are called the coefficient of degree i of the polynomial
- Two polynomials are equal if, and only if, for every $i \in \mathbb{N}$, they have the same coefficient of degree i
- If $m \in \mathbb{N}$ is the largest integer such that $a_m \neq 0$, we say that it is the degree of the polynomial

Notation. The set of all polynomials with coefficients in R is written $R[X]$.

- $2 + 5X$: polynomial in $\mathbb{Z}[X]$ of degree 1
- $5X + 2$: same polynomial, order of the terms does not matter
- $0X^2 + 5X + 2$: same polynomial, terms with 0 coefficient don't matter.

Note: polynomials work over every ring! This is a polynomial with coefficients in $\mathbb{R}^{2\times 2}$:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix} X^2 + \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Definition. Let $(R, +, \times)$ be a ring. A polynomial with coefficients in R is an expression of the form
$$a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m$$
where $a_0, \ldots, a_m \in R$.

- a_i are called the coefficient of degree i of the polynomial
- Two polynomials are equal if, and only if, for every $i \in \mathbb{N}$, they have the same coefficient of degree i
- If $m \in \mathbb{N}$ is the largest integer such that $a_m \neq 0$, we say that it is the degree of the polynomial

Notation. The set of all polynomials with coefficients in R is written $R[X]$.

- $2 + 5X$: polynomial in $\mathbb{Z}[X]$ of degree 1
- $5X + 2$: same polynomial, order of the terms does not matter
- $0X^2 + 5X + 2$: same polynomial, terms with 0 coefficient don't matter.

Enumerate all the polynomials of degree ≤ 2 with coefficients in $\mathbb{Z}/2\mathbb{Z}$.
How many are there?

- 1
- 2
- 4
- 8
- infinitely many

Definition. Let $(R, +, \times)$ be a ring. A polynomial with coefficients in R is an expression of the form

$$a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m$$

where $a_0, \ldots, a_m \in R$.

- a_i are called the coefficient of degree i of the polynomial
- Two polynomials are equal if, and only if, for every $i \in \mathbb{N}$, they have the same coefficient of degree i
- If $m \in \mathbb{N}$ is the largest integer such that $a_m \neq 0$, we say that it is the degree of the polynomial

Notation. The set of all polynomials with coefficients in R is written $R[X]$.

- $2 + 5X$: polynomial in $\mathbb{Z}[X]$ of degree 1
- $5X + 2$: same polynomial, order of the terms does not matter
- $0X^2 + 5X + 2$: same polynomial, terms with 0 coefficient don't matter.

Implementation: a polynomial $A \in R[X]$ is just implemented as an array `A` where `A[i]` is the coefficient of degree i.

$(X^2+2X+2)+(X^3+X+1) = X^3+X^2+3X+1$ (things you already probably know)

$(X^2+2X+2)\times(X^3+X+1) = X^5+2X^4+3X^3+3X^2+4X+2$

In general:

Definition. Let $A = a_0 + a_1X + \cdots + a_dX^d$ and $B = b_0 + b_1X + \cdots + b_dX^d$. Define

$$A + B = (a_0 + b_0) + (a_1+b_1)X + \cdots + (a_d+b_d)X^d$$

and

$$A \times B = a_0b_0 + (a_0b_1 + a_1b_0)X + \cdots + a_db_dX^{2d}$$

Different operations! On different sets!

(Coefficient of degree i in $A\times B$ is $\sum_{j=0}^{i} a_jb_{i-j}$)

What happens to the degree?

- $\deg(A+B) \leq \max(\deg(A), \deg(B))$
- $\deg(A\times B) \leq \deg(A) + \deg(B)$
- $\deg(A\times B) = \deg(A) + \deg(B)$ if coefficients in a field (we define $\deg(0) = -\infty$ for this to be true)

$(X^2 + 2X + 2) + (X^3 + X + 1) = X^3 + X^2 + 3X + 1$ (things you already probably know)

$(X^2 + 2X + 2) \times (X^3 + X + 1) = X^5 + 2X^4 + 3X^3 + 3X^2 + 4X + 2$

In general:

Definition. Let $A = a_0 + a_1X + \cdots + a_dX^d$ and $B = b_0 + b_1X + \cdots + b_dX^d$. Define

$$A + B = (a_0 + b_0) + (a_1 + b_1)X + \cdots + (a_d + b_d)X^d$$

and

$$A \times B = a_0b_0 + (a_0b_1 + a_1b_0)X + \cdots + a_db_dX^{2d}$$

Different operations! On different sets!

(Coefficient of degree i in $A \times B$ is $\sum_{j=0}^{i} a_jb_{i-j}$)

Theorem. Let R be a ring. Then $(R[X], +, \times)$ is a ring.

Theorem. Let $a, b \in \mathbb{Z}$ with $a \neq 0$. There exists a unique pair of integers q, r such that:

- $b = qa + r$
- $r \in \{0, \ldots, |a| - 1\}$

The first item makes sense if we see a, b, q, r as polynomials, but the second does not.

Theorem. Let $\mathbb{K}$ be a field. Let $A, B \in \mathbb{K}[X]$ with $A \neq \mathbf{0}$.
There exists a unique pair $Q, R \in \mathbb{K}[X]$ of polynomials such that:

- $B = QA + R$
- $\deg(R) < \deg(A)$.

what is called `divmod` in Python

Definition. A divides B if $R = 0$

```
def divmod(B,A):
  assert(A != 0)
  Q,R = 0,B
  while deg(R) >= deg(A):
    a,r = A[deg(A)],R[deg(R)]
    S = (r/a) * X**(deg(R)-deg(A))
    Q = Q+S
    R = R - S*A
  return (Q,R)
```

$$
\begin{array}{r|l}
X^6 + 3X^5 + 7X^4 + 7X^3 + 6X^2 + 3 & X^2 + 2X + 3 \\
-(X^6 + 2X^5 + 3X^4) & \hline \\
\hline
X^5 + 4X^4 + 7X^3 + 6X^2 + 3 & X^4 + X^3 + 2X^2 \\
-(X^5 + 2X^4 + 3X^3) & \\
\hline
2X^4 + 4X^3 + 6X^2 + 3 & \\
-(2X^4 + 4X^3 + 6X^2) & \\
\hline
3 &
\end{array}
$$

$$X^6 + 3X^5 + 7X^4 + 7X^3 + 6X^2 + 3 = (X^4 + X^3 + 2X^2)(X^2 + 2X + 3) + 3$$

Theorem. Let $\mathbb{K}$ be a field. Let $A, B \in \mathbb{K}[X]$ with $A \neq \mathbf{0}$.
There exists a unique pair $Q, R \in \mathbb{K}[X]$ of polynomials such that:

- $B = QA + R$
- $\deg(R) < \deg(A)$.

what is called `divmod` in Python

Definition. A divides B if $R = 0$

```
def divmod(B,A):
  assert(A != 0)
  Q,R = 0,B
  while deg(R) >= deg(A):
    a,r = A[deg(A)],R[deg(R)]
    S = (r/a) * X**(deg(R)-deg(A))
    Q = Q+S
    R = R - S*A
  return (Q,R)
```

We need $\mathbb{K}$ to be a field! Otherwise $1/a$ $(= a^{-1})$ does not necessarily exist.

Example. $A = 2, B = X$ polynomials in $\mathbb{Z}[X]$.

- Suppose $B = QA + R$, where $Q, R \in \mathbb{Z}[X]$
- Then $X = (q_0 + q_1 X)2 = 2q_0 + 2q_1 X$
- So $1 = 2q_1$

Definition. Let $A, B \in \mathbb{K}[X]$. We say that $D \in \mathbb{K}[X]$ is a gcd of A and B if:

- D divides A and B
- Every divisor of A and B has degree at most $\deg(D)$

(note: it is not unique. If D is a gcd, then $2D$ is also a gcd)

Can be computed using Euclid's algorithm!
Also get Bézout's coefficients directly from this

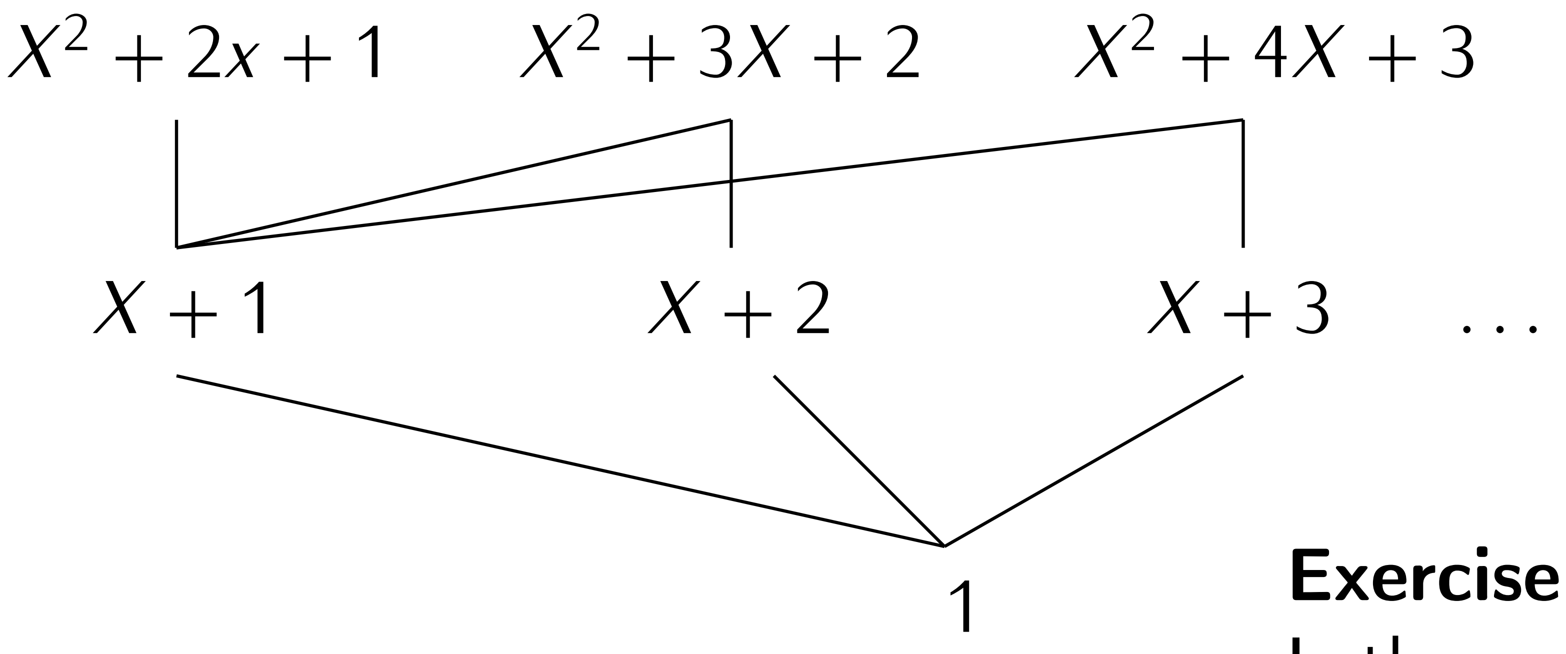
```
def euclid(A,B):
  if deg(A) > deg(B):
    A,B = B,A # swap A and B
  if A == 0:
    return B

  remainders = [B,A]
  while remainders[-1] != 0:
    B = remainders[-2]
    A = remainders[-1]
    Q,R = divmod(B,A)
    remainders.append(R)

  return remainders[-2]
```

$$\gcd(X^2 - X - 6, X^2 + 3X + 2) = -4X - 8 = -4(X + 2)$$

$$X^2 - X - 6 = 1 \cdot (X^2 + 3X + 2) + (-4X - 8)$$

$$X^2 + 3X + 2 = (-1/4X - 1/4)(-4X - 8) + 0$$

- "A divides B" is an order on $\mathbb{K}[X]$
- can define prime polynomials just like for numbers
- can prove the existence/uniquess of prime decompositions
- there are infinitely many prime polynomials
- can define an equivalence relation $\equiv_D$ for $D \in \mathbb{R}[X]$
- ...



Exercise Think about these things!
Is the polynomial $X^2 + 1$ in $\mathbb{R}[X]$ prime? Why/why not?

Summary

- Rings "ultimate" structures that generalizes the notions you know about addition/multiplication
- Commutative rings with division = fields
- Polynomials with coefficients in a field behave a lot like $\mathbb{Z}$ (division, gcd, primes, . . .)
- fields/polynomials pop up all the time in CS

Next week

- A bit more about finite fields
- Application: error-correcting codes (QR codes, space communication, . . .)

Final week

- Final exam organisation
- Recap of notions
- Quizzes