Discrete Algebraic Structures

WiSe 2025/2026

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Cheaters beware!

We know some students cheated at the quizzes in some of the previous weeks

You have a choice:

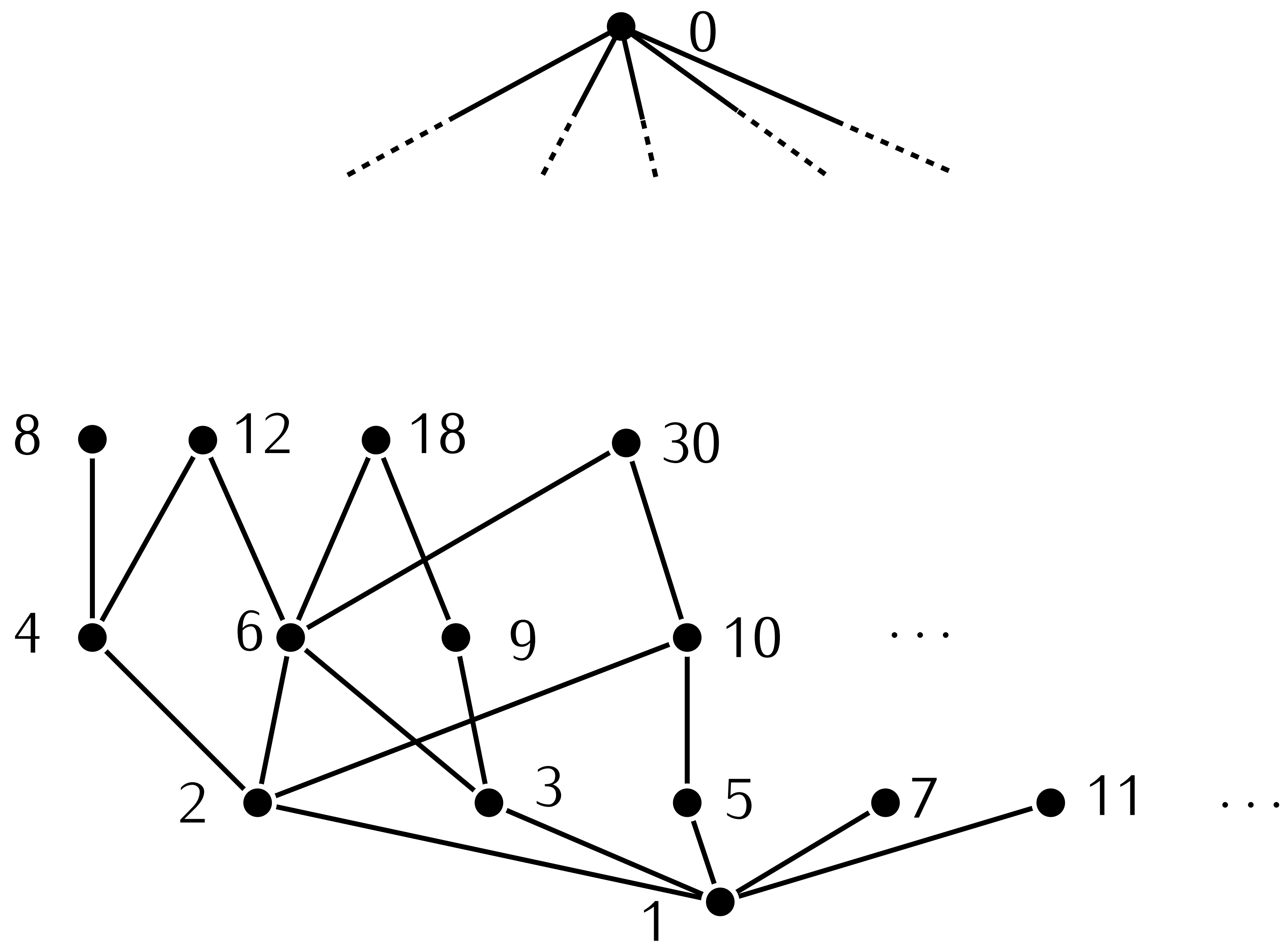
- Option 1:
  - Send an email to me (antoine.wiehe@tuhh.de) and Michel Sénégas (michel.senegas@tuhh.de) to confess
  - You list **all** the weeks in which you cheated. If you “forget” one, see Option 2.
  - You will not get the bonus points, but you don’t get reported
- Option 2: get reported for cheating at the Examination Office

In case you do not reply by Wednesday 1pm, we will consider that you have decided for Option 2.

Definition. Let $S \subseteq \mathbb{N}_0$, and $d \in \mathbb{N}_0$. We say:

- d is a common divisor of S if for all $s \in S$, we have d divides s
- d is a greatest common divisor of S if it is a common divisor and for every common divisor d' of S, we have d' divides d

We write $a \wedge b$ for the greatest common divisor (gcd) of $\{a, b\}$.



Theorem. For any $a, b \in \mathbb{N}_0$, the number e given by `euclid` is $\gcd(a, b)$.

Two things to prove:

- e divides a and b
- every d that divides a and b also divides e

```
def euclid(a,b):
  if a > b:
    a,b = b,a # swap a and b
  if a == 0:
    return b

  remainders = [b,a]
  while remainders[-1] != 0:
    b = remainders[-2]
    a = remainders[-1]
    q,r = divmod(b,a)
    remainders.append(r)

  return remainders[-2]
```

Lemma. If d divides a and b, then d divides every element of `remainder`.

Proof by induction: `remainders` $= (r_0, r_1, \ldots, r_k, 0)$
For every $i \in \{0, \ldots, k\}$, d divides r_i
True for $i = 0, 1$ by assumption
For $i \geq 2$:

$$r_{i-2} = q \cdot r_{i-1} + r_i$$

divisible by d
$r_{i-2} = d \cdot x$

divisible by d
$(r_{i-1} = d \cdot y)$

So $r_i = d \cdot (x - qy)$ is divisible by d. $\square$

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Lemma. e divides a and b.

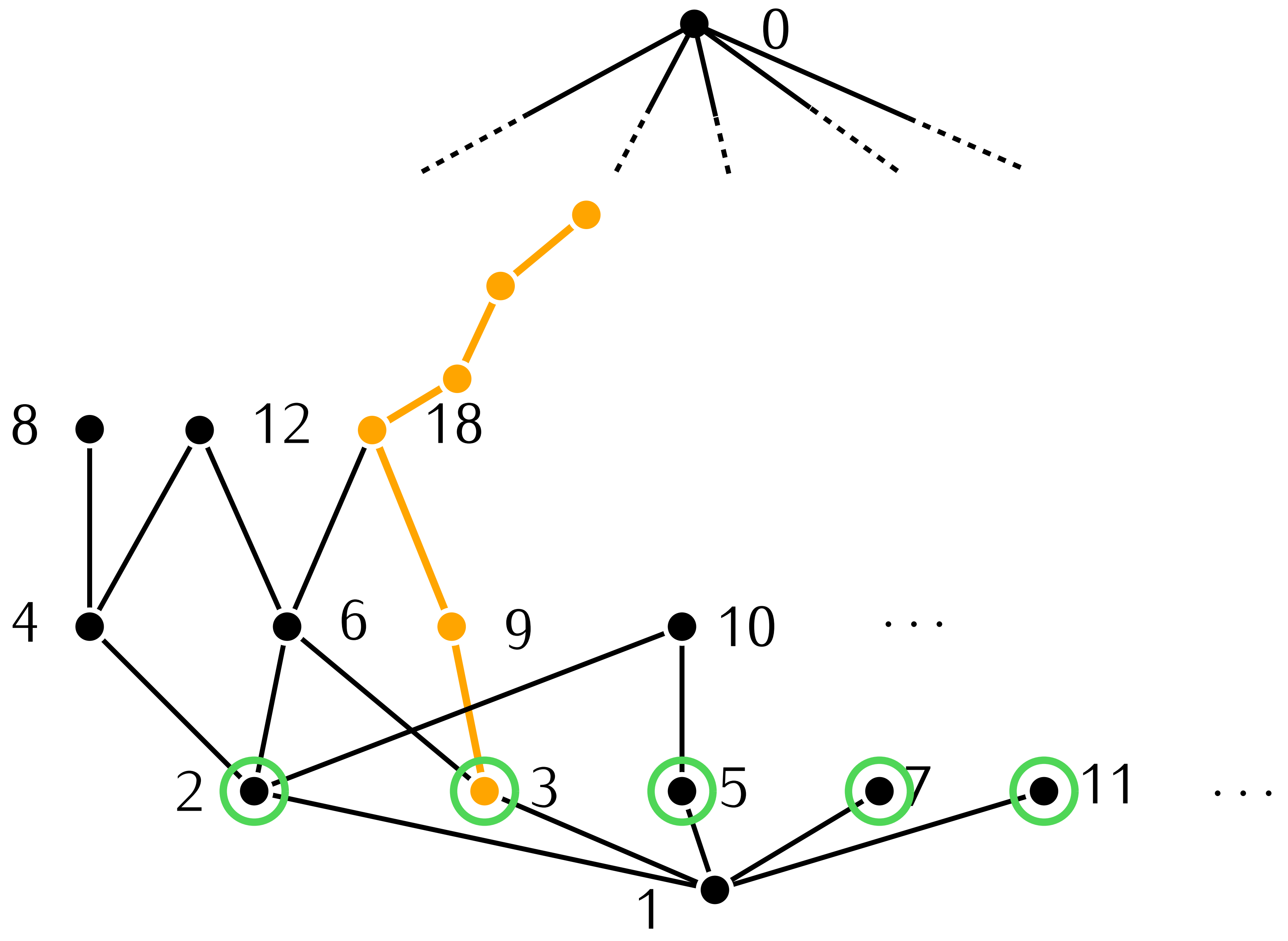
$$
\begin{aligned}
b &= q_2 \times a + r_2 && \Rightarrow e \text{ divides } b \\
a &= q_3 \times r_2 + r_3 && \Rightarrow e \text{ divides } a \\
r_2 &= q_4 \times r_3 + r_4 && \Rightarrow e \text{ divides } r_2 \\
&\vdots \\
r_{k-2} &= q_k \times r_{k-1} + r_k && \Rightarrow e \text{ divides } r_{k-2} \\
r_{k-1} &= q_{k+1} \times r_k + 0 && \Rightarrow e \text{ divides } r_{k-1}
\end{aligned}
$$

this is e

Prime numbers

Definition. A number $p \in \mathbb{N}_0$ is prime if it is > 1 and it is only divisible by 1 and itself.

Theorem. Every number $n \in \mathbb{N}$ that is not 1 is divisible by a prime number.



(*) Being prime can be defined for $p \in \mathbb{Z}$ as well

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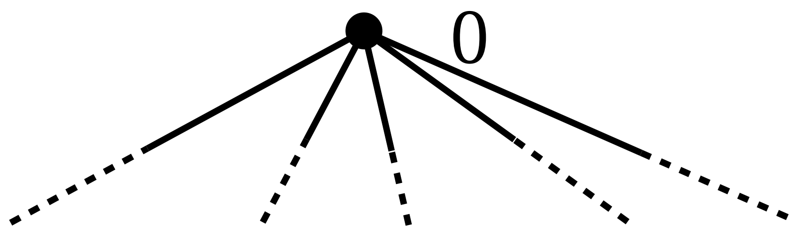
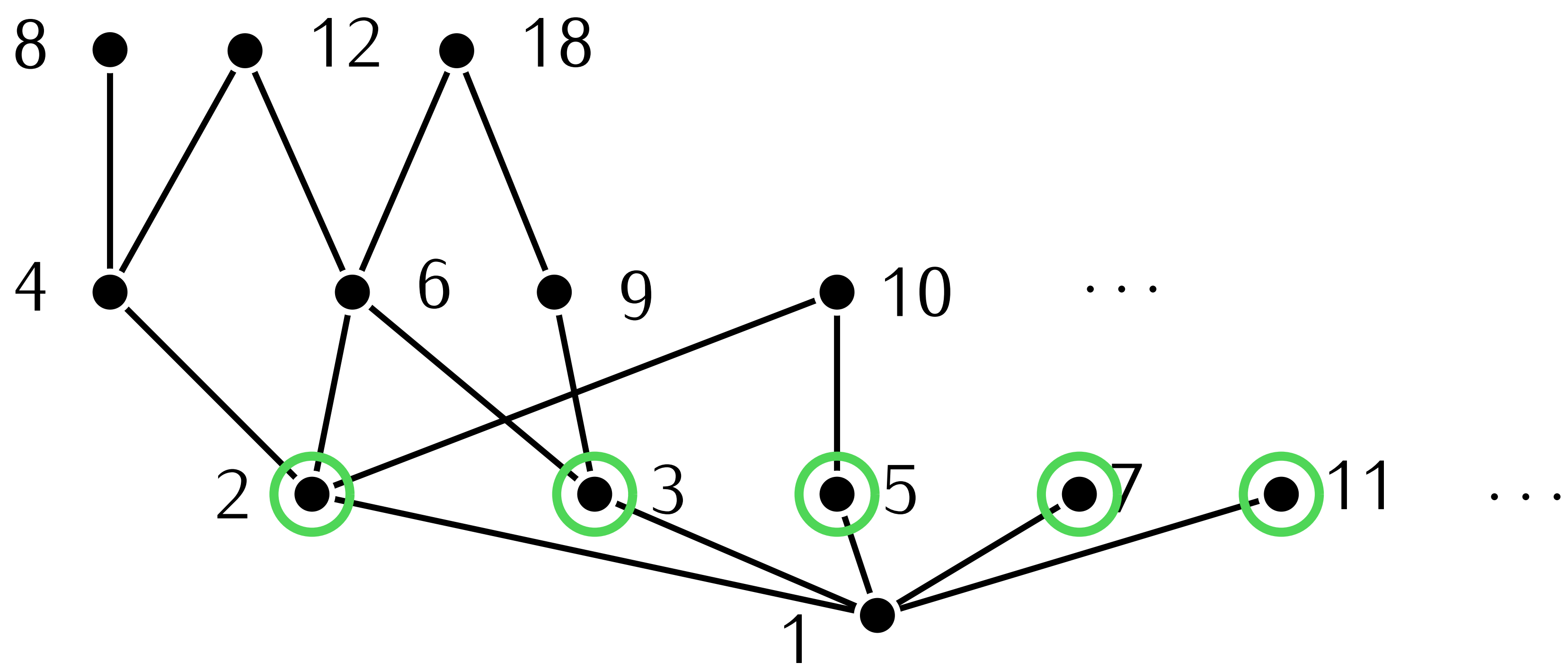
Theorem. There are infinitely many prime numbers.



- Let $p_1, \ldots, p_k$ be any list of prime numbers.
- Define $N = p_1 \ldots p_k + 1$
- N is divisible by a prime p



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- Let $p_1, \ldots, p_k$ be any list of prime numbers.
- Define $N = p_1 \ldots p_k + 1$
- N is divisible by a prime p
- $N = (p_1 \ldots p_{k-1})p_k + 1$ and $1 < p_k$, so p is not p_k
- Similarly for every $i \in \{1, \ldots, k-1\}$, N is not divisible by p_i.
  So $p \notin \{p_1, \ldots, p_k\}$
- $p_1, \ldots, p_k$ cannot be a list of all the primes $\square$

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Definition. A number $p \in \mathbb{N}_0$ is prime if it is > 1 and it is only divisible by 1 and itself.

Theorem. Let p be a prime number and $a, b \in \mathbb{Z}$. If p divides ab, then it divides a or it divides b.

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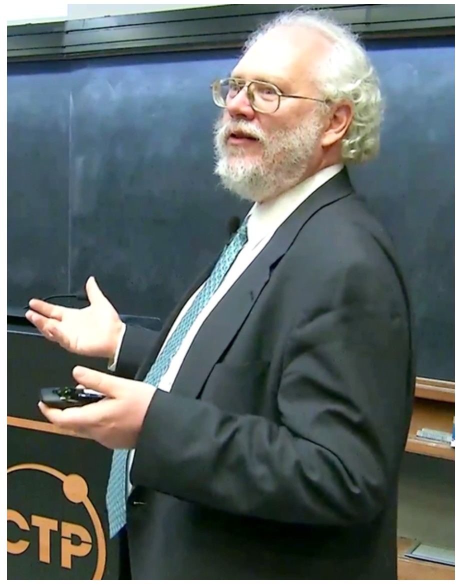
Theorem. Let $n \in \mathbb{N}$ be such that $n \geq 2$.
There exist prime numbers $p_1 < \cdots < p_k$ and $e_1, \ldots, e_k \in \mathbb{N}$ such that $n = p_1^{e_1} \cdots p_k^{e_k}$.
Moreover this decomposition is unique.

- Important in math (building blocks for all numbers)
  There are still many open questions about prime numbers (Goldbach's conjecture, Twin Prime conjecture, Riemann's hypothesis)
- Important in crypto (RSA)
- It seems hard to compute $p_1, \ldots, p_k$ if given $n \in \mathbb{N}$
- **Theoretically,** there is a quantum algorithm that can do this and **break RSA**.

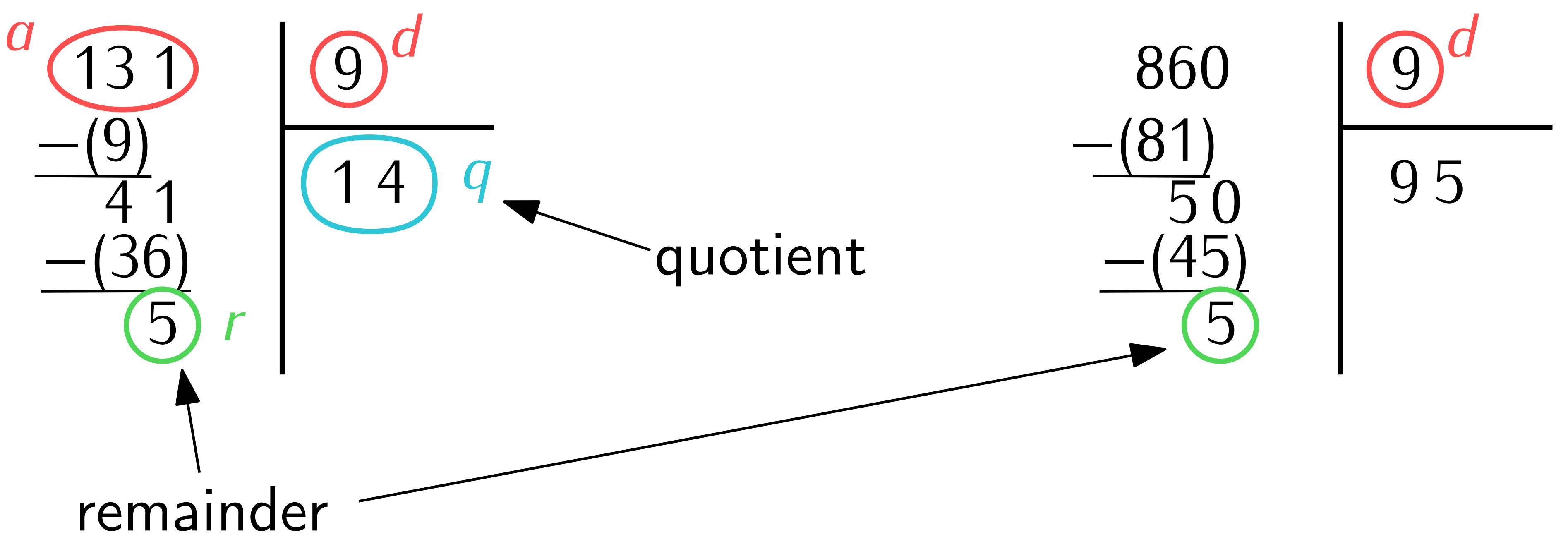


Peter Shor

Modular arithmetic

Theorem. For all $a, d \in \mathbb{Z}$ such that $a \neq 0$, there exists a unique pair $(q, r) \in \mathbb{Z}^2$ such that:

- $a = q \cdot d + r$
- $r \in \{0, \ldots, |d| - 1\}$



Almost the same as divmod in Python:

```
divmod(131,9) # (14,5)
divmod(10,-3) # (-4,-2)
```

- If remainder is 0: we say d divides a
  a is a multiple of d

Notation: $d \mid a$

- Define $a \equiv_d b$ by "they have the same remainder in the division by d"
  $131 \equiv_9 860$

Definition. Let $d \geq 2$. Define $a \equiv_d b$ by "b and b' have the same remainder in the division by d"
We then say that a and b are congruent modulo d.

Equivalently: $a \equiv_d b$ if $a - b$ is divisible by d

Notation. Other common notations for the same thing: $a \equiv b \bmod d$ or $a = b \bmod d$

Methods: to check if $1075 \equiv 364 \bmod 72$

Compute $1075 - 364 = 711$
Check if 711 is divisible by 72

Divide 1075 by 72: $1075 = 14 \times 72 + 67$
Divide 364 by 72: $364 = 5 \times 72 + 4$
Check if the remainders are equal

Which of the following are true?

- $14 \equiv 6 \bmod 2$
- $3 \equiv 1 \bmod 4$
- $61 \equiv 5 \bmod 7$

Congruence modulo d

Antoine Wiehe

Definition. Let $d \geq 2$. Define $a \equiv_d b$ by "b and b' have the same remainder in the division by d"
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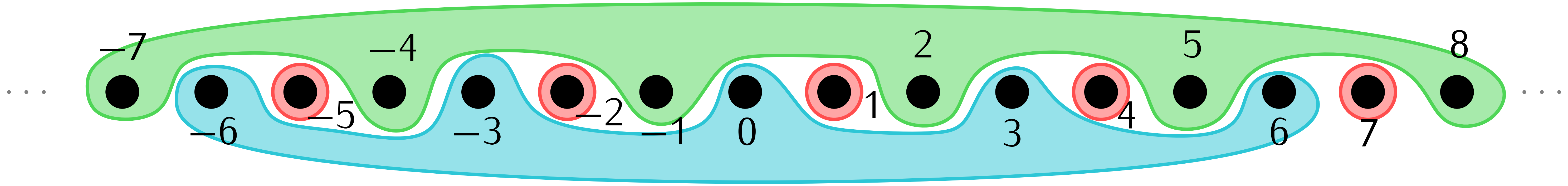
Theorem. $\equiv_d$ is an equivalence relation.

Equivalence relation	Equivalence class	Set of equivalence classes
$\equiv_d$	$\cancel{[5]_{\equiv_d}}$ $[5]_d$	$\cancel{\mathbb{Z}/\equiv_d}$ $\mathbb{Z}/d\mathbb{Z}$

There are exactly d equivalence classes: $|\mathbb{Z}/d\mathbb{Z}| = d$



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Examples.
- $d = 12$: $\mathbb{Z}/d\mathbb{Z}$ is $\{[1]_{12}, [2]_{12}, \ldots, [12]_{12}\}$ hours on a clock
- $d = 7$: $\{[1]_7, [2]_7, \ldots, [7]_7\}$ days of the week
- $d = 2$: $\{[0]_2, [1]_2\}$ parity

Algebra: study of **operations** and **equations** on *stuff*

Number theory

Boolean algebra

Numbers: $+, \times, 1/x, 1, 0$

Matrices: $+, \times, M^{-1}, I, 0$

Sets: $\cap, \cup, \times, \Delta, \emptyset, \ldots$

Functions: $\circ, f^{-1}, \mathrm{Id}_A$

Booleans: $\wedge, \vee, \Rightarrow, \neg, \top, \bot$

Relations: $\circ, R^T, \mathrm{Id}, \cup, \cap, \times, \ldots$

Linear algebra

Relational algebra

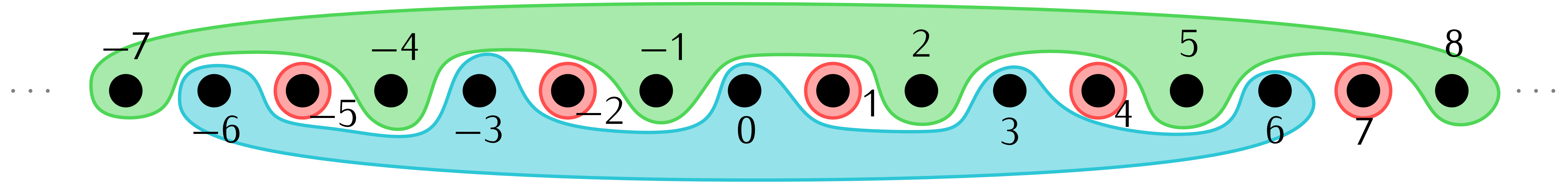
Modular integers: $+, \times, [a]_d^{-1}, [1]_d, [0]_d$

Arithmetic = study of $+$ and $\times$ over $\mathbb{N}_0$ or $\mathbb{Z}$

Modular Arithmetic = study of $+$ and $\times$ over $\mathbb{Z}/d\mathbb{Z}$

Definition. We define addition and multiplication on $\mathbb{Z}/d\mathbb{Z}$ as follows:

- $[a]_d + [b]_d$ is defined to be $[a+b]_d$
- $[a]_d \times [b]_d$ is defined to be $[a \times b]_d$



Note that this is a **definition** and it depends on d:

$\boxed{\{\dots,-1,2,5,\dots\}} + \boxed{\{\dots,-3,0,3,\dots\}} =$ $\boxed{\{\dots,-1,2,5,\dots\}} + \boxed{\{\dots,-2,1,4,\dots\}} =$

The usual rules of addition and multiplication are true in modular arithmetic:

- $[a]_d + [b]_d = [b]_d + [a]_d$ and $[a]_d \times [b]_d = [b]_d \times [a]_d$
- $[0]_d + [a]_d = [a]_d$
- $[a]_d \times ([b]_d + [c]_d) = [a]_d \times [b]_d + [a]_d \times [c]_d$

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Which day of the week will it be in one week?

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~~Which day of the week will it be in one week?~~
Which day of the week will it be in one month (January 15, 2026)?

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~~Which day of the week will it be in one week?~~
~~Which day of the week will it be in one month (January 15, 2026)?~~
Which day of the week was it two months ago (October 15, 2025)?

We have defined addition and multiplication in $\mathbb{Z}/d\mathbb{Z}$.
Can we define subtraction and division?

- Subtraction: $[a]_d - [b]_d = [a-b]_d$, easy
- Division: $[b]_d/[a]_d = \cancel{[b/a]_d}$?!

The usual division that we know (for numbers in $\mathbb{Q}$, for example) satisfies: $\frac{b}{a} = b \times \boxed{\frac{1}{a}}$ and $a \times \boxed{\frac{1}{a}} = 1$

Definition. Let $a \in \mathbb{Z}$. An inverse of a modulo d is a number $b \in \mathbb{Z}$ such that $a \times b \equiv_d 1$.

Definition (Reminder (?) from your Math 1 course)**.** Let M be an $n \times n$ matrix. An inverse of N is a matrix such that $MN = I_n$.

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Definition. Let $a \in \mathbb{Z}$. An inverse of a modulo d is a number $b \in \mathbb{Z}$ such that $a \times b \equiv_d 1$.

Examples.
- 2 is an inverse of 2 modulo 3
- 3 is an inverse of 2 modulo 5
- inverse of 2 modulo 4?

$\gcd(a, d) = 1$

Theorem. Let $a \in \mathbb{Z}$ and $d \geq 2$. Then a has an inverse modulo d if, and only if, a and d are coprime.

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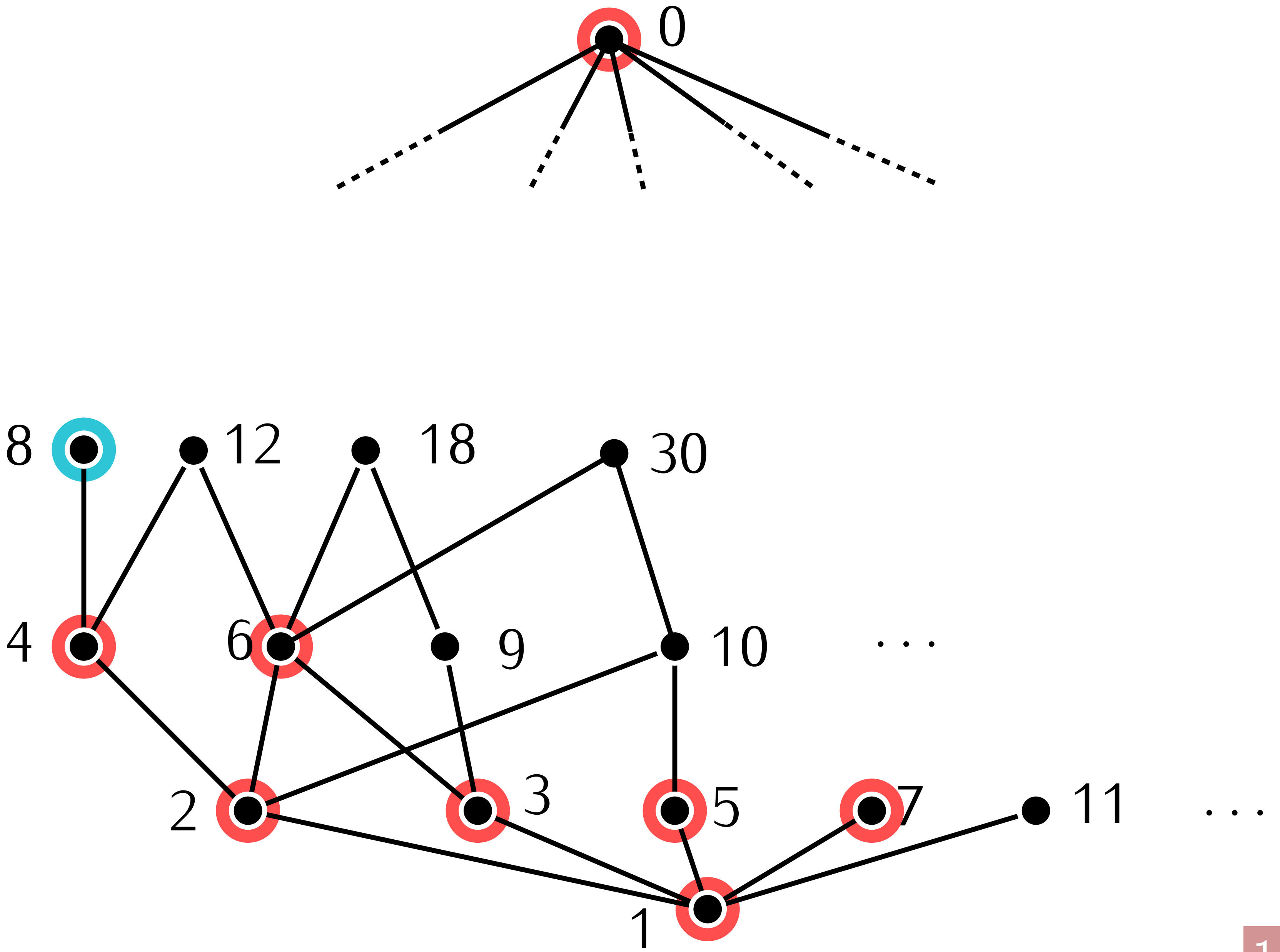
Which of the numbers in $\{0, \ldots, 7\}$ have an inverse modulo 8?

There are 4 numbers that have an inverse



Definition. Let $n \geq 1$. Define $\varphi(n)$ to be the number of numbers in $\{0, \ldots, n-1\}$ that have an inverse modulo n.

$$\varphi(n) = |\{a \in \{0, \ldots, n-1\} \mid \gcd(a, n) = 1\}|$$

n	1	2	3	4	5	6	7	8	9
	$\{0\}$	$\{1\}$	$\{1,2\}$	$\{1,3\}$	$\{1,2,3,4\}$	$\{1,5\}$	$\{1,2,3,4,5,6\}$	$\{1,3,5,7\}$	$\{1,2,4,5,7,8\}$
$\varphi(n)$	1	1	2	2	4	2	6	4	6

Theorem. Let n have prime decomposition $p_1^{e_1} \times \cdots \times p_k^{e_k}$. Then $\varphi(n) = \prod_{i=1}^{k} (p_i - 1)p_i^{e_i - 1}$.

Examples.
- $12 = 2^2 3$ so $\varphi(12) = (1 \times 2^1)(2 \times 3^0) = 4$
- $125 = 5^3$ so $\varphi(125) = 4 \times 5^2 = 100$

What is $\varphi(60)$?

Theorem. Let a and d be such that $\gcd(a,d)=1$. Then $a^{\varphi(d)} \equiv 1 \bmod d$.

So $a \times a^{\varphi(d)-1} = a^{\varphi(d)} \equiv 1 \bmod d$

Proof: need some extra algebraic tools that we will see in a couple lectures

Application: compute large powers modulo a number

What are the last 2 digits of 7^{582} (when written in base 10)?

$\log_{10}(7^{582}) = 582\log_{10}(7) \approx 491.847\ldots$

$$7^{582} = (d_{491}d_{490}\ldots d_2\underline{d_1d_0})_{10} = d_{491}10^{491} + \cdots + d_3 \times 1000 + d_2 \times 100 + d_1 \times 10 + d_0 = \sum_{i=0}^{491} d_i 10^i$$

what we want to know: $7^{582} \equiv (d_1 \times 10 + d_0) \bmod 100$