Automata Theory and Formal Languages

Summer Term 2026

4th Lecture

Chomsky Hierarchy

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Chomsky Hierarchy

Containment hierarchy of classes of formal grammars. Grammars are classified into four different types with different limitations

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no restrictions

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Type 1

Each rule $w_1 \rightarrow w_2$ satisfies $|w_1| \leq |w_2|$ with the exception that $S \rightarrow \varepsilon$ is allowed if S does not occur on any right hand side of the rules.
$|w_1| \leq |w_2|$ implies that no shortening rules are allowed.

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Type 2 (context free)

Same restriction as type 1 and additionally for each rule $w_1 \rightarrow w_2$ the string w_1 contains only a single variable (i.e. $w_1 \in V$)

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Type 3 (regular)

Same restriction as type 2 and additionally $w_2 \in \Sigma \cup \Sigma V$, i.e. the right hand side is either a single terminal symbol or a terminal symbol followed by a variable.

Chomsky Hierarchy

Remark

The order of the variable and terminal symbol is chosen arbitrarily. Important is just that within a regular grammar only a single order occurs. If for all rules $w_2 \in \Sigma \cup V\Sigma$ it would also be a regular grammar.

Chomsky Hierarchy

Definition

A language $L \subset \Sigma^*$ is said to be of type 0 (1,2,3) if there exist a type 0 (1,2,3) grammar G for which $L(G) = L$.

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In order to show that a language L is of type 0 (1,2,3) one thus just has to find a corresponding grammar.

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Remark

In order to show that a language L is of type 0 (1,2,3) one thus just has to find a corresponding grammar. In order to show that a grammar is not of type 0 (1,2,3) one has to prove that no type 0 (1,2,3) grammar can exist generating the corresponding language.

Chomsky Hierarchy

Remark

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- For *context-sensitive languages* rules of the form

$$uAv \to uxv$$

are allowed, i.e. A can only be replaced by x if it occurs between a u and a v.

Chomsky Hierarchy

Difference between type 0 and type 1 languages:

For type 0 languages the string length can decrease during the derivation process.

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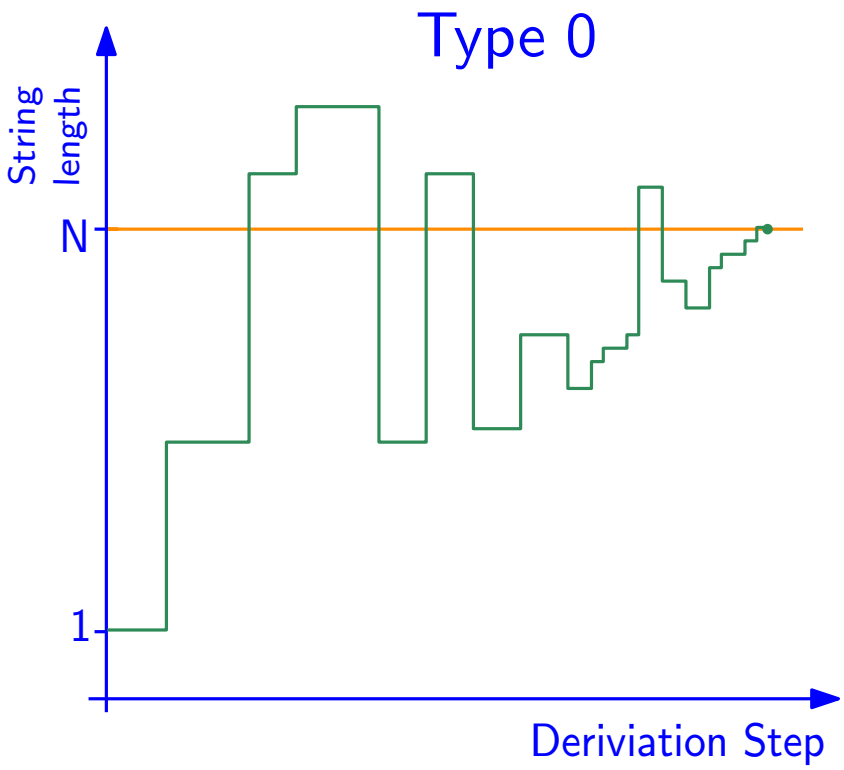

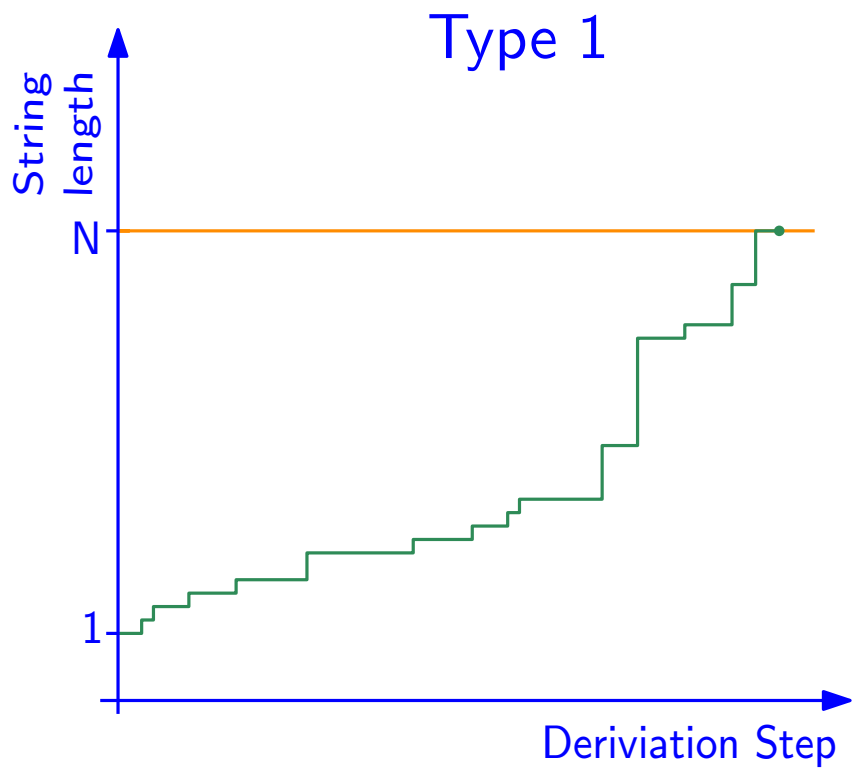

Special ε rule

In some text books the special ε rule $S \to \varepsilon$ is not included for type 1 grammars. In that case the languages cannot generate the empty string. We include it directly in the Chomsky hierarchy to avoid this.

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For type 2/3 grammar one can additionally allow rules of the form $A \to \varepsilon$ where A is any variable. This does not change the power of the language class.

Containment hierarchy of Chomsky Hierarchy

type 3 languages $\subsetneq$ type 2 languages $\subsetneq$ type 1 languages $\subsetneq$ type 0 languages

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Difference sets

- type 2 but not type 3: $L = \{a^n b^n \mid n \geq 0\}$

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- type 1 but not type 2: $L = \{a^n b^n c^n \mid n \geq 0\}$

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type 3 languages $\subsetneq$ type 2 languages $\subsetneq$ type 1 languages $\subsetneq$ type 0 languages

Difference sets

- type 2 but not type 3: $L = \{a^n b^n \mid n \geq 0\}$
- type 1 but not type 2: $L = \{a^n b^n c^n \mid n \geq 0\}$
- type 0 but not type 1:
  $L = \{w \mid w \text{ describes a terminating Turing machine}\}$

Decision Problem

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given: String x of length n and grammar G

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Problem: Is $x \in L(G)$?

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Theorem

The decision problem for type 1 (2,3) languages is decidable (computable).

Decision Problem

Proof

If $x \in L(G)$ there exist a derivation

$$S \Rightarrow x_1 \Rightarrow x_2 \Rightarrow \cdots \Rightarrow x_k = x.$$

We consider the shortest existing derivation (there may be multiple).
Since G is of type 1 the string length does never decrease.

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By systematically trying out all derivation branches an algorithm can determine if the grammar G can generate the string x.

The algorithm either terminates if all intermediate strings during the derivation process have reached a length of $n + 1$ or if the string x has been generated.

Decision Problem

Definition

The set of all strings of length $\leq n$ that can be derived in m steps can be inductively defined as

$$T_0^n := \{S\}$$
$$T_{m+1}^n := T_m^n \cup \{w' \in (\Sigma \cup V)^* \mid |w'| \leq n \wedge (\exists w'' \in T_m^n)[w'' \Rightarrow w']\}$$

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Since $|T_m^n| \leq \sum_{i=1}^{n} |\Sigma \cup V|^i$ there exists an m_0 so that

$$T_{m_0}^n = T_{m_0+1}^n = T_{m_0+2}^n = \cdots =: T^n$$

Decision Problem

Algorithm for the decision problem

$n := |w|$
$T := \{S\}$
$T' := \emptyset$
while $T \neq T'$ **do**
  $T' := T$
  $T := T \cup \{w' \in (\Sigma \cup V)^* \mid |w'| \leq n \wedge (\exists w'' \in T')[w'' \Rightarrow w']\}$
end while
if $w \in T$ **then**
  return true
else
  return false
end if

Decision Problem

Remark

The algorithm requires exponential time, and the decision problem is NP-hard.

Backus-Naur Form

Formalism for the expression of context-free (type 2) grammars (short BNF form)

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Was used when the programming language ALGOL60 was designed

Backus-Naur Form

For rules with the same left hand side

$$
\begin{aligned}
A &\rightarrow \beta_1 \\
A &\rightarrow \beta_2 \\
&\vdots \\
A &\rightarrow \beta_n
\end{aligned}
$$

Backus-Naur Form

For rules with the same left hand side

$$A \to \beta_1$$
$$A \to \beta_2$$
$$\vdots$$
$$A \to \beta_n$$

one can use the meta rule

$$A \to \beta_1 | \beta_2 | \ldots | \beta_n.$$

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(Backus and Naur actually used ::= instead of $\to$)

Extended Backus-Naur Form (EBNF)

additionally

Optional string

$$A \to \alpha[\beta]\gamma$$

for

$$A \to \alpha\gamma$$
$$A \to \alpha\beta\gamma$$

Extended Backus-Naur Form (EBNF)

and

Repetition

$$A \to \alpha\{\beta\}\gamma$$

for

$$A \to \alpha\gamma$$
$$A \to \alpha B\gamma$$
$$B \to \beta$$
$$B \to \beta B$$

Extended Backus-Naur Form (EBNF)

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Repetition

$$A \to \alpha\{\beta\}\gamma$$

for

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$$A \to \alpha B\gamma$$
$$B \to \beta$$
$$B \to \beta B$$

Meaning

β can occur arbitrary often between α and γ (also not at all)