Automata Theory and Formal Languages

Summer Term 2026

20th Lecture

The Myhill-Nerode Theorem

Prof. Dr. Matthias Mnich

Institute for Algorithms and Complexity

Myhill-Nerode Relations

Let Σ be an alphabet. Let $L \subseteq \Sigma^*$ be a language Let $\equiv$ be an equivalence relation on Σ^*. Then $\equiv$ is a Myhill-Nerode relation for L if it satisfies:

1. It is right congruent, i.e. for all $x, y \in \Sigma^*$ and all $a \in \Sigma$, if $x \equiv y$, then $xa \equiv ya$.
2. It refines L, i.e. if $x \equiv y$, then $x \in L \Leftrightarrow y \in L$.
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Note: $\equiv$ is an equivalence relation on the strings in Σ^*. (The relation $\approx$ in a previous lecture was an equivalence relation on the states of a DFA.) Even if the number of equivalence

classes of $\equiv$ is finite, the size of each class need not be finite. (At least one of them must contain an infinite number of strings since Σ^* is infinite.)

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Construction a relation from an automaton

Let M be a DFA over an alphabet Σ with start state q_0 such that all states are reachable from q_0. Define the relation $\equiv_M$ on Σ^* such that $x \equiv_M y$ if $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$.

Then $\equiv_M$ is an equivalence relation on Σ^*. We will show that $\equiv_M$ is also a Myhill-Nerode relation for $L(M)$.

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1. Let x and y be arbitrary strings in Σ^* and let a be an arbitrary symbol in Σ. Assume $x \equiv_M y$. Then $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$ by definition. We get $\hat{\delta}(q_0, xa) = \delta(\hat{\delta}(q_0, x), a) = \delta(\hat{\delta}(q_0, y), a) = \hat{\delta}(q_0, ya)$.

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2. Let x and y be arbitrary strings in Σ^*. Assume $x \equiv_M y$. Then $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$. Obviously, M either accepts both x and y, or M rejects both x and y, so $x \in L(M)$ if and only if $y \in L(M)$. That is, $\equiv_M$ refines $L(M)$.

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3. For each $x \in \Sigma^*$, the equivalence class of x is $[x] = \{y \in \Sigma^* | x \equiv_M y\} = \{y \in \Sigma^* \mid \hat{\delta}(q_0, y) = \hat{\delta}(q_0, x)\}$.

Let $Q = \{q_0, ..., q_n\}$ be the states of M. For each $i = 0, ..., n$, let $x_i \in \Sigma^*$ be a string such that $\hat{\delta}(q_0, x_i) = q_i$. (Such a string must exist since we assume all states are accessible from q_0.) Then, $[x_i] \neq [x_j]$ for all $i \neq j$, i.e. there is an equivalence class $[x_i]$ for each state $q_i \in Q$.

Suppose there is a string $y \in \Sigma^*$ s.t. $[y] \neq [x_i]$ for all $i = 0, ..., n$. Then $\hat{\delta}(q_0, y) \neq q_i$ for all $q_i \in Q$. This is impossible, so $\equiv_M$ must have exactly $n + 1$ equivalence classes.

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Converse: construct automaton from relation

Let Σ be an alphabet and $L \subseteq \Sigma^*$ a language.
Suppose $\equiv$ is a Myhill-Nerode relation for L. Then $\equiv$ has a finite number of equivalence relations, so we can construct a DFA $M_\equiv = (Q, \Sigma, \delta, q_0, E)$ for L as follows:

- $Q = \{[x] \mid x \in \Sigma^*\}$,
- $q_0 = [\varepsilon]$,
- $E = \{[x] \mid x \in L\}$,
- $\delta([x], a) = [xa]$.

Then $L(M_\equiv) = L$ (see Schöning's book for a proof).

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The previous two constructions give the following result: Let L be a language over some alphabet. Then L is regular if and only if there is a Myhill-Nerode relation for L.

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Automata isomorphism

Let $M = (Q^M, \Sigma, \delta^M, q_0^M, E^M)$ and $N = (Q^N, \Sigma, \delta^N, q_0^N, E^N)$ be two DFAs. Then M and N are isomorphic if there exists a bijective function $f : Q_M \to Q_N$ such that

1. $f(q_0^M) = q_0^N$,
2. $f(\delta^M(p, a)) = \delta^N(f(p), a)$ for all $p \in Q^M$ and $a \in \Sigma$,
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That is we can rename the states of M so it becomes identical to N.

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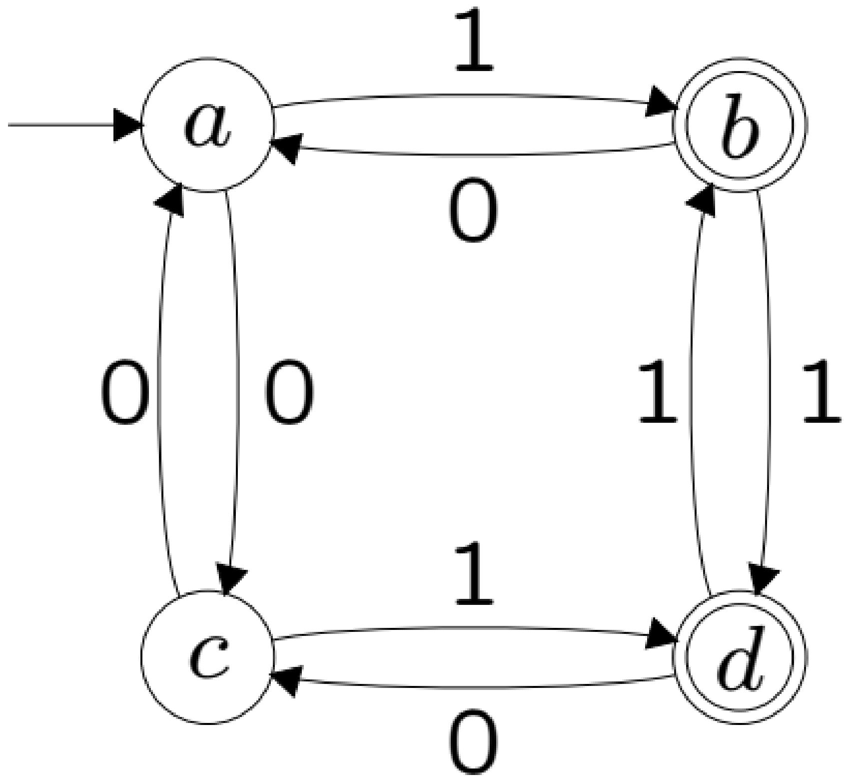

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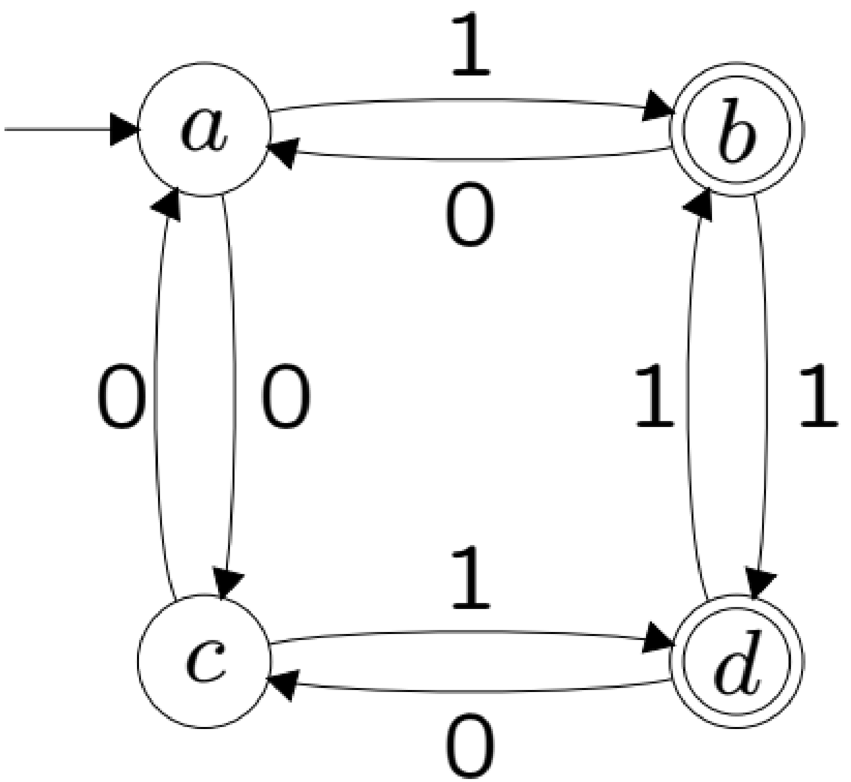

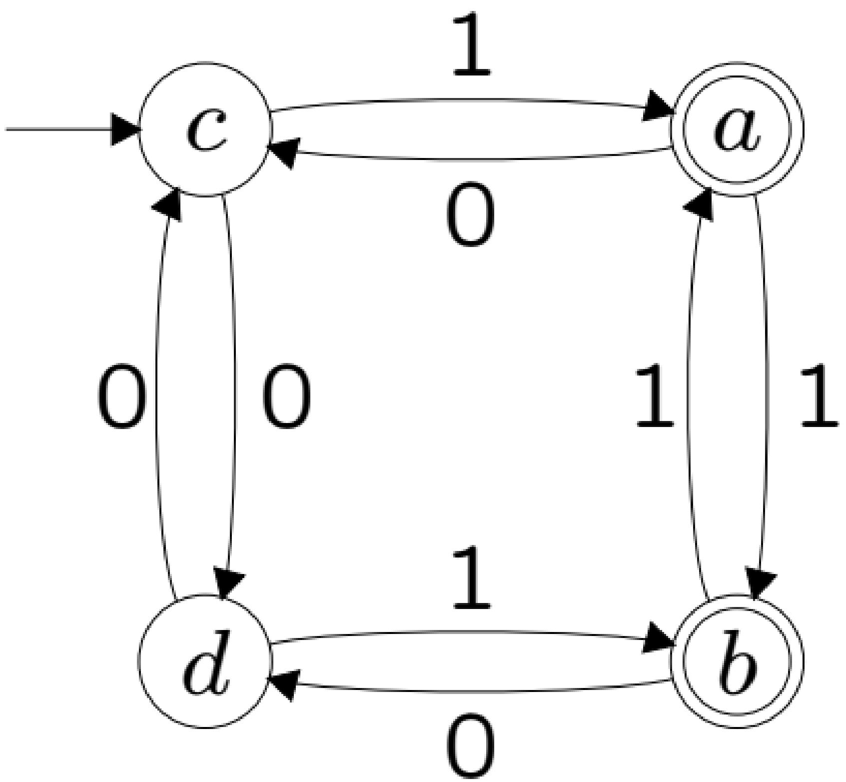

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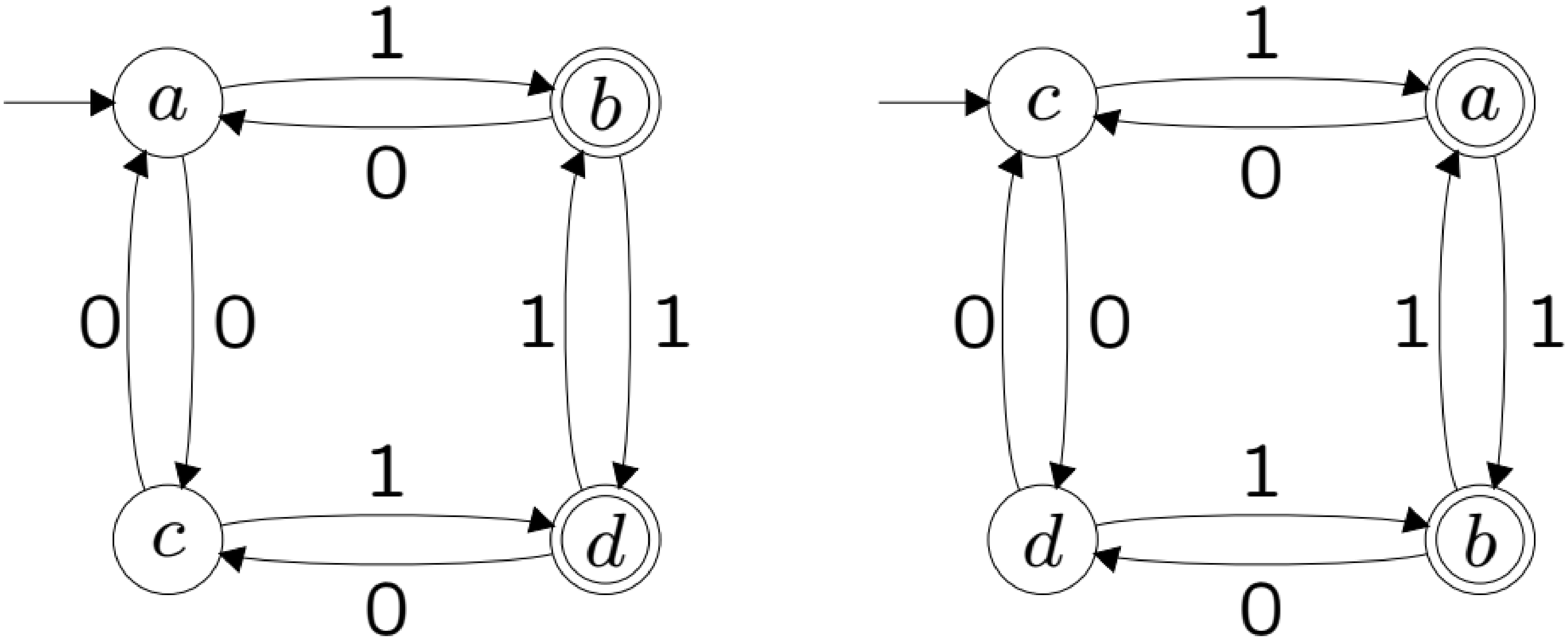

A closer analysis of the constructions

We have shown two constructions:

- $M \mapsto \equiv_M$: Given a DFA, construct a Myhill-Nerode relation $\equiv_M$.
- $\equiv \mapsto M_{\equiv}$: Given a Myhill-Nerode relation, construct a DFA.

These constructions are inverses of each other in the following sense.

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Let L be a regular language with a Myhill-Nerode relation $\equiv$.

1. Construct the DFA $M_\equiv$ for $\equiv$.
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The Myhill-Nerode Theorem

Recall that a relation on Σ^* is a subset of $\Sigma^* \times \Sigma^*$, i.e. a set of pairs of strings. Let $\equiv_1$ and $\equiv_2$ be two equivalence relations on Σ^*. Then $\equiv_1$ refines $\equiv_2$ if $\equiv_1 \subseteq \equiv_2$, i.e. if $x \equiv_1 y \Rightarrow x \equiv_2 y$).

We say that $\equiv_1$ is finer than $\equiv_2$ and that $\equiv_2$ is coarser than $\equiv_1$.

The finest possible relation is $\{(x, x) \mid x \in \Sigma^*\}$. The coarsest possible relation is $\{(x, y) \mid x, y \in \Sigma^*\}$.

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