Discrete Algebraic Structures

WiSe 2025/2026

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What to know: functions

- Definition; be able to determine if a given set $f \subseteq A \times B$ is a function or not
- Properties of functions: injectivity, surjectivity, bijectivity
- Composition of functions
- Identity function
- Inverses

Introduction to Logic

- A language: a syntax (how to write things) and a semantics (what the things mean)
- Propositional logic and Predicate logic

Propositional logic:

- the "mathematics of 0 and 1"
- Applications in CS engineering: down to the hardware side, everything is 0/1. Logic important for:
  - electronic circuit synthesis
  - minimization of circuits
- Applications in math: how to prove things
  - 0=False,
  - 1=True
  - how to manipulate mathematical statements without making reasoning mistakes
  - how to recognize fallacies also in everyday life

Predicate logic:

- In CS: automated software verification
  (increasingly employed and demanded at tech giants like Microsoft and Amazon)
- In math: more powerful language using quantifiers

Syntax = what is legal to write

Definition (Propositional formulas)**.** The following are the only legal formulas:

1. every variable $p, q, r, \ldots$ is a formula
2. $\top$ and $\bot$ are formulas, (TRUE and FALSE)
3. If ϕ and ψ are formulas, then $\phi \wedge \psi$ is a formula, (AND)
4. If If ϕ and ψ are formulas, then $\phi \vee \psi$ is a formula, (OR)
5. If ϕ and ψ are formulas, then $\phi \Rightarrow \psi$ is a formula, (IMPLIES)
6. If ϕ is a formula, then $\neg\phi$ is a formula. (NOT)

Example. $p \vee q$, $p \vee \top$, $p \wedge p$, $p \wedge \wedge p$, $\neg p \wedge \neg(\neg p)$, $(\neg p) \wedge ((\neg q) \Rightarrow r)$, $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

formula (by 1.)

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formula
(by 4.)

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formula
(by 1.)

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(by 2.)

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formula
(by 3.)

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we are allowed to put parentheses/brackets around formulas,
like $3x + 1$ and $3x + (1)$
like $x + y \times z$ and $(x + y) \times z$

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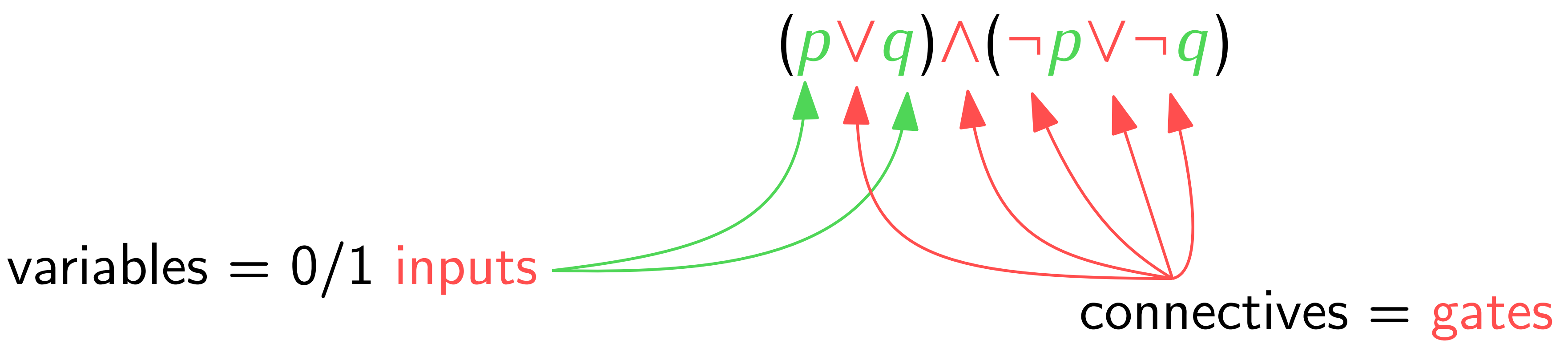
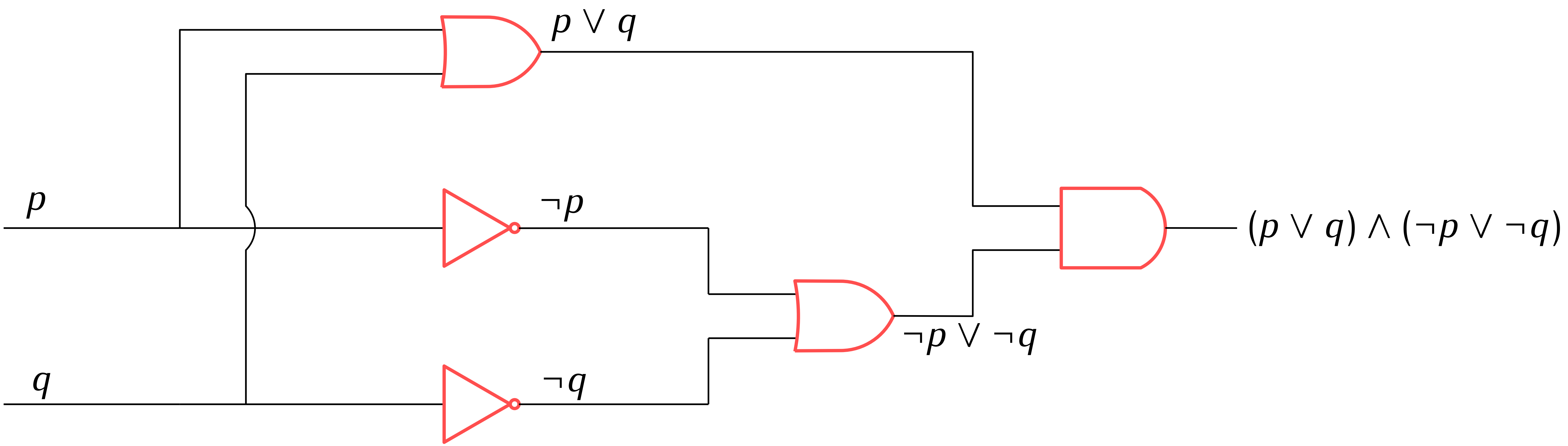
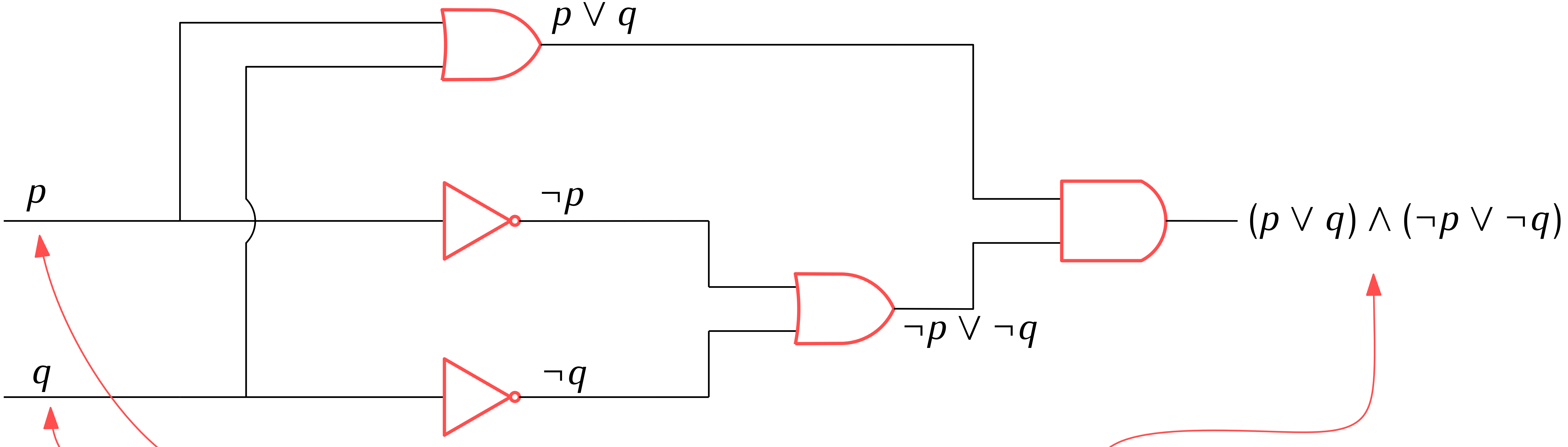
Which of the remaining examples is a legal formula?





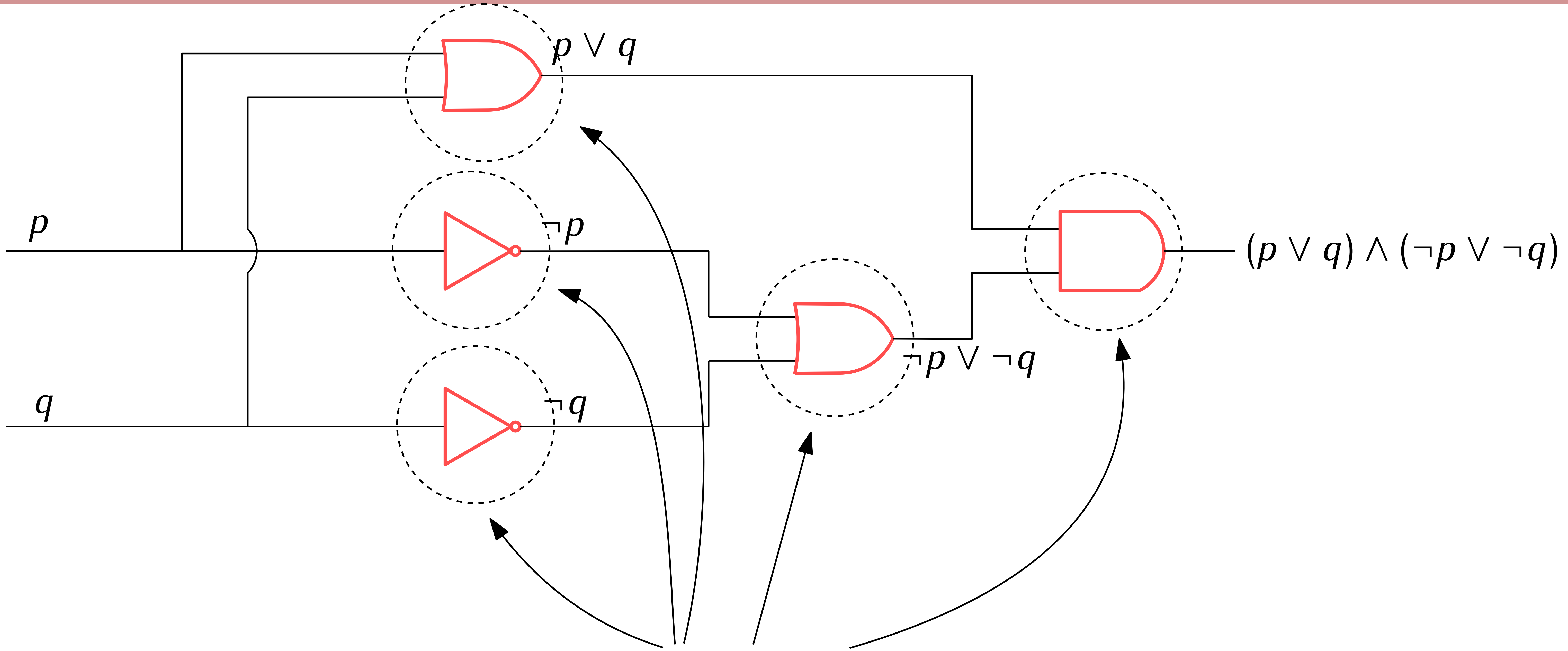
Question. For which values of the inputs (0/1) does the circuit output 1?



For each value of these in $\{0, 1\}$, we have a value of this in $\{0, 1\}$

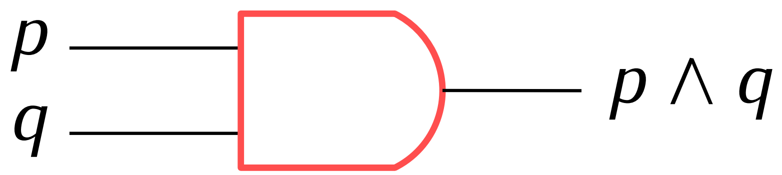
p	q	$(p \lor q) \land (\neg p \lor \neg q)$
0	0	
0	1	
1	0	
1	1	

truth table of $(p \lor q) \land (\neg p \lor \neg q)$



We start by defining the truth table for these

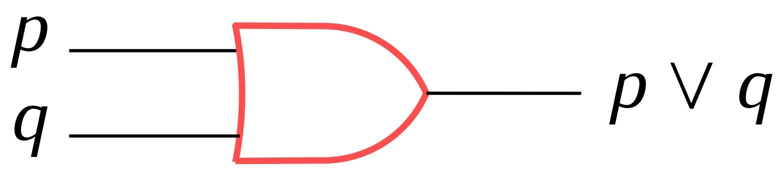
- Case of $\wedge$:



p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

both inputs must be 1

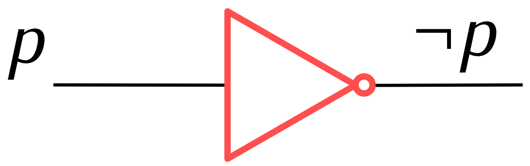
- Case of $\vee$:



p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

it suffices that one of the inputs is 1

- Case of $\neg$:

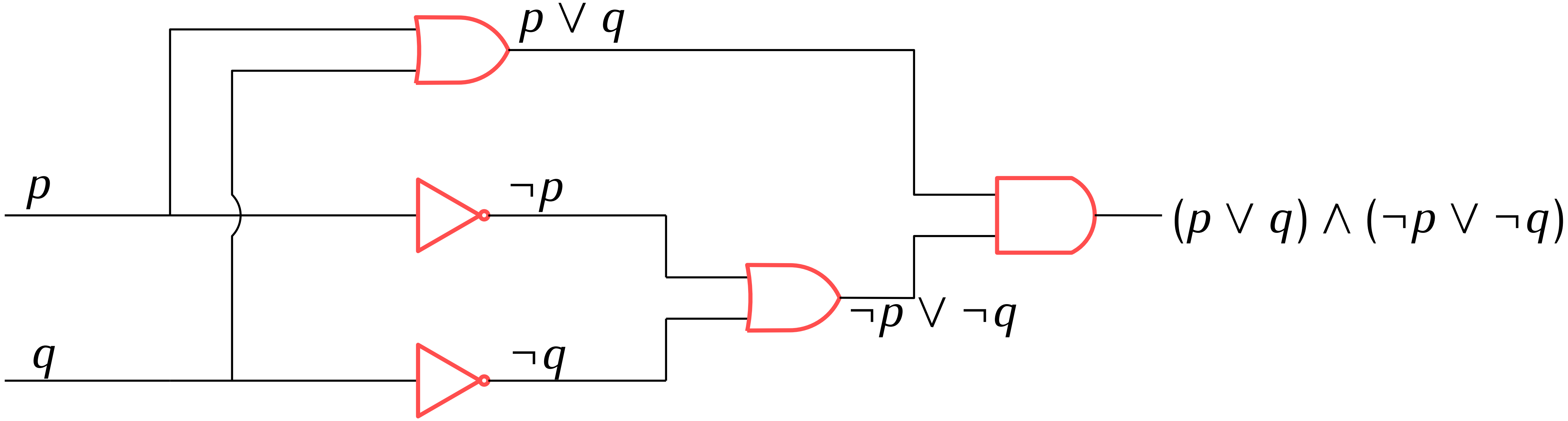


p	$\neg p$
0	1
1	0

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

p	$\neg p$
0	1
1	0

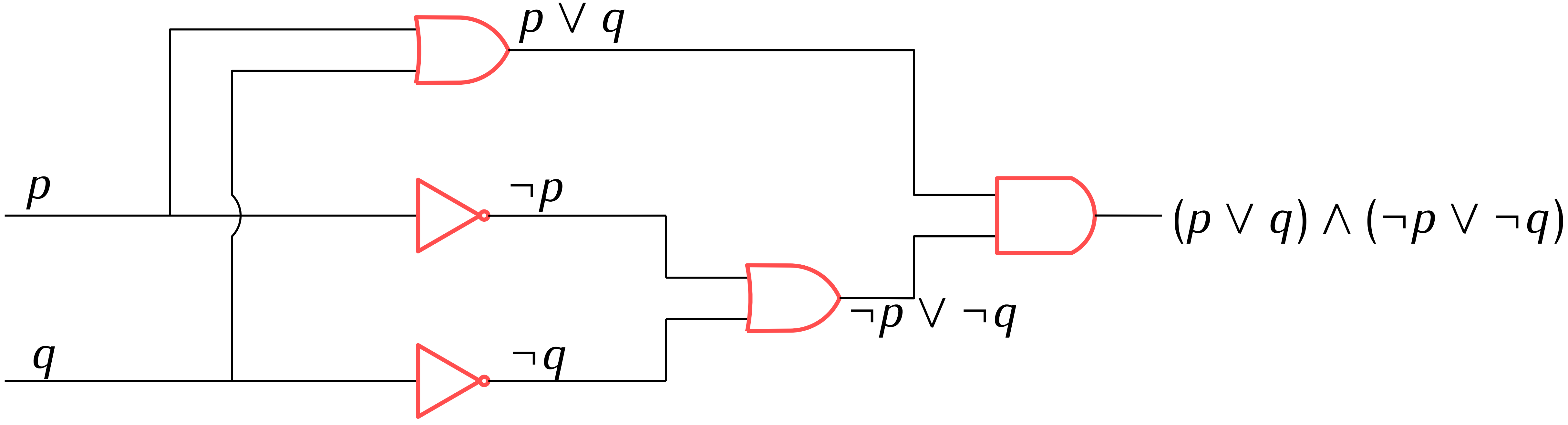


p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	1				
0	1	1				
1	0	0				
1	1	0				

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

p	$\neg p$
0	1
1	0

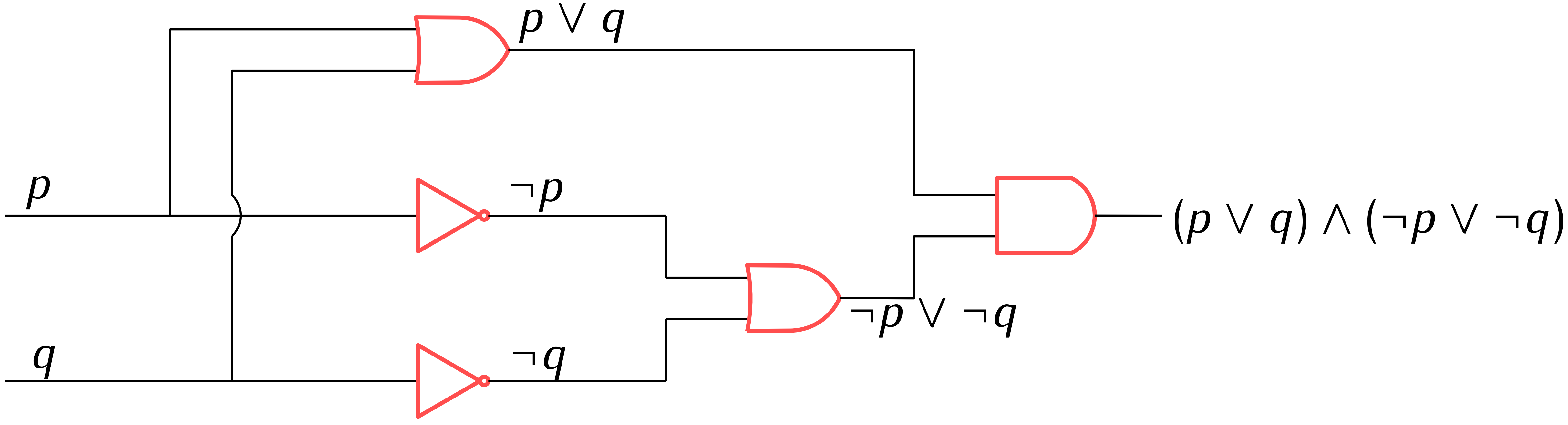


p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	1	1			
0	1	1	0			
1	0	0	1			
1	1	0	0			

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

p	$\neg p$
0	1
1	0

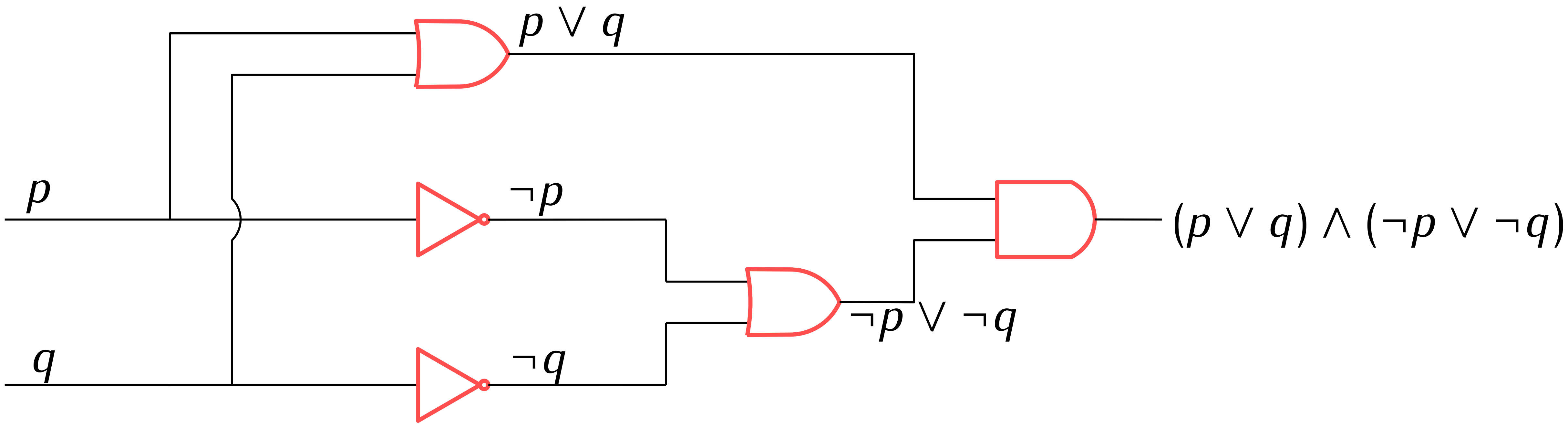


p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	1	1	0		
0	1	1	0	1		
1	0	0	1	1		
1	1	0	0	1		

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

p	$\neg p$
0	1
1	0

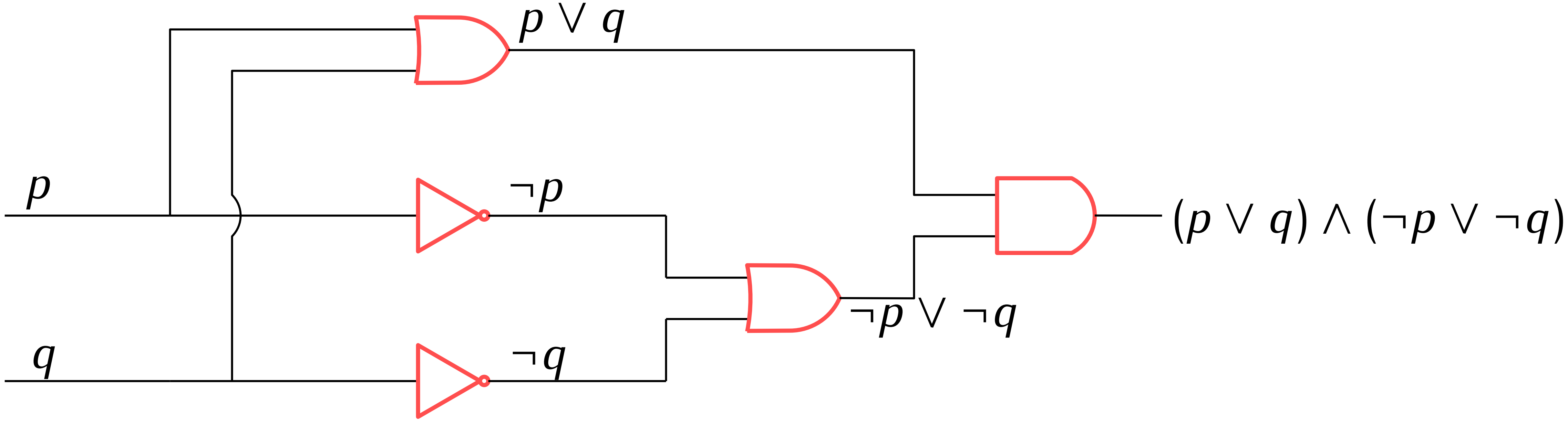


p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	1	1	0	1	
0	1	1	0	1	1	
1	0	0	1	1	1	
1	1	0	0	1	0	

p	q	$p \land q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \lor q$
0	0	0
0	1	1
1	0	1
1	1	1

p	$\neg p$
0	1
1	0



p	q	$\neg p$	$\neg q$	$p \lor q$	$\neg p \lor \neg q$	$(p \lor q) \land (\neg p \lor \neg q)$
0	0	1	1	0	1	0
0	1	1	0	1	1	1
1	0	0	1	1	1	1
1	1	0	0	1	0	0

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no truth table defined yet!

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

- "p implies q" means "whenever p is true, then q is true"
- We also say "If p then q".
- This is wrong only when p is true and q is false

Which of these statements are true?

- If cats can fly, the Earth is flat.
- If $0 = 1$, then $\cos$ is injective.
- If $1 = 1$, then $\sin$ is injective.

Definition. Let φ be a formula. A satisfying assigment for φ is a way to evaluate the variables/input so that φ becomes true.

Satisfying assignment "=" row in truth table that finishes with 1

satisfying assignments for $(p \vee q) \wedge (\neg p \vee \neg q)$

$\{(0,1),(1,0)\}$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	1	1	0	1	0
0	1	1	0	1	1	1
1	0	0	1	1	1	1
1	1	0	0	1	0	0

To find the satisfying assignments of a formula: no better way than building the truth table.
Problem with a \$1'000'000 reward: find a faster method to even find one satisfying assignment.

Definition. Two formulas φ and ψ are logically equivalent if they have **exactly** the same satisfying assignments.

We write: $\varphi \equiv \psi$

Standard equivalences to know (some proofs in HÜ):

$$\varphi \wedge (\psi \vee \theta) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \theta)$$
$$\varphi \vee (\psi \wedge \theta) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \theta)$$

Distributivity laws

$$\varphi \wedge (\psi \wedge \theta) \equiv (\varphi \wedge \psi) \wedge \theta$$
$$\varphi \vee (\psi \vee \theta) \equiv (\varphi \vee \psi) \vee \theta$$

Associativity laws

$$\varphi \wedge \top \equiv \varphi$$
$$\varphi \vee \bot \equiv \varphi$$

Identity laws

Theorem. The following formulas are logically equivalent:

$$\left.\begin{aligned} \neg\neg\varphi &\equiv \varphi \\ \neg(\varphi \land \psi) &\equiv \neg\varphi \lor \neg\psi \\ \neg(\varphi \lor \psi) &\equiv \neg\varphi \land \neg\psi \end{aligned}\right\}\text{De Morgan's rules}$$

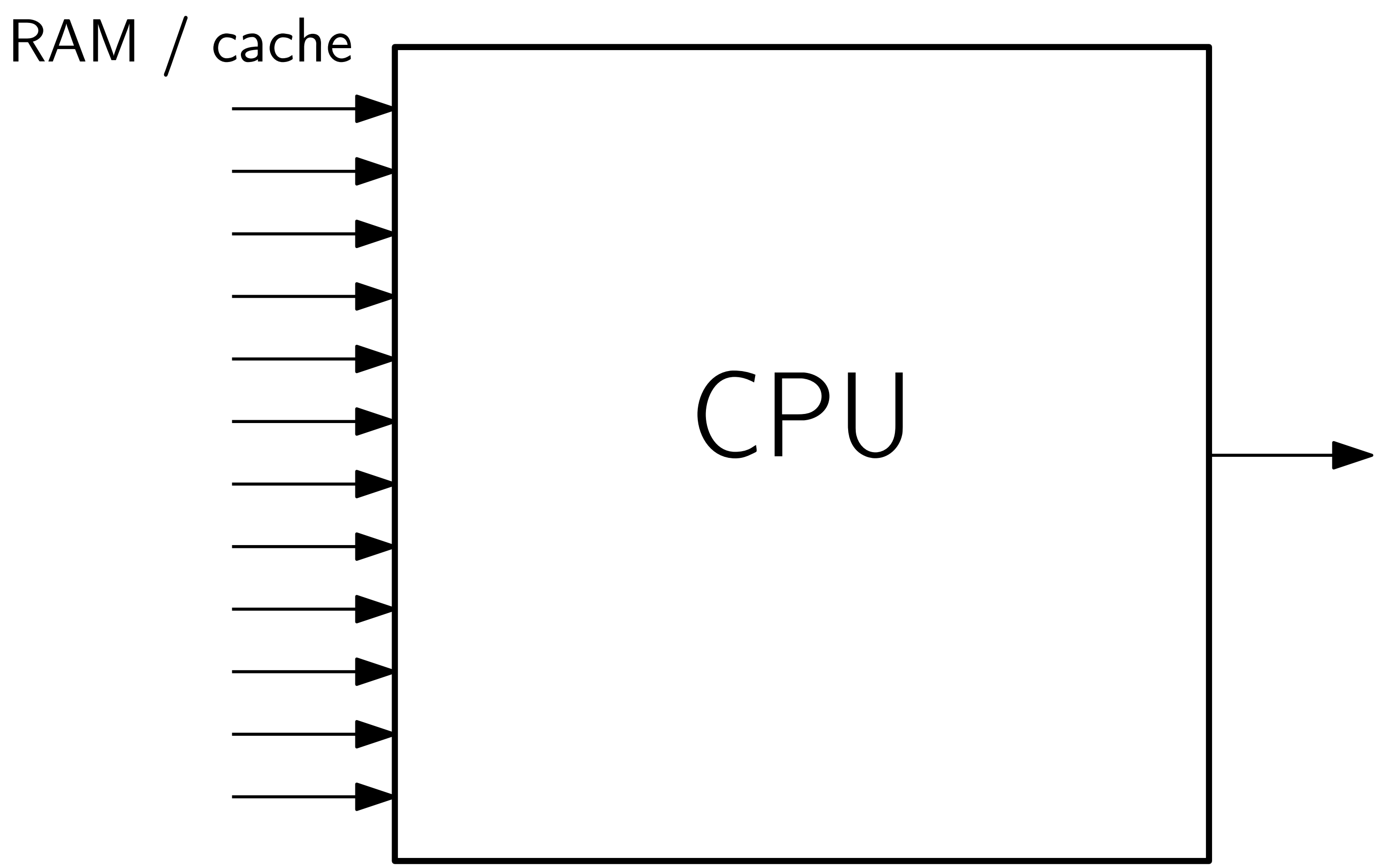
φ	$\neg\varphi$	$\neg\neg\varphi$
0	1	0
1	0	1

φ	ψ	$\neg\varphi$	$\neg\psi$	$\neg\varphi \lor \neg\psi$	$\neg(\varphi \land \psi)$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

φ	ψ	$\neg\varphi$	$\neg\psi$	$\neg\varphi \land \neg\psi$	$\neg(\varphi \lor \psi)$
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

- We have seen how to compute a truth table, given any formula
- Can one do the reverse? "Given a truth table, compute a formula"

Concrete application: modelize the behavior of any digital system using logic/electronic circuits



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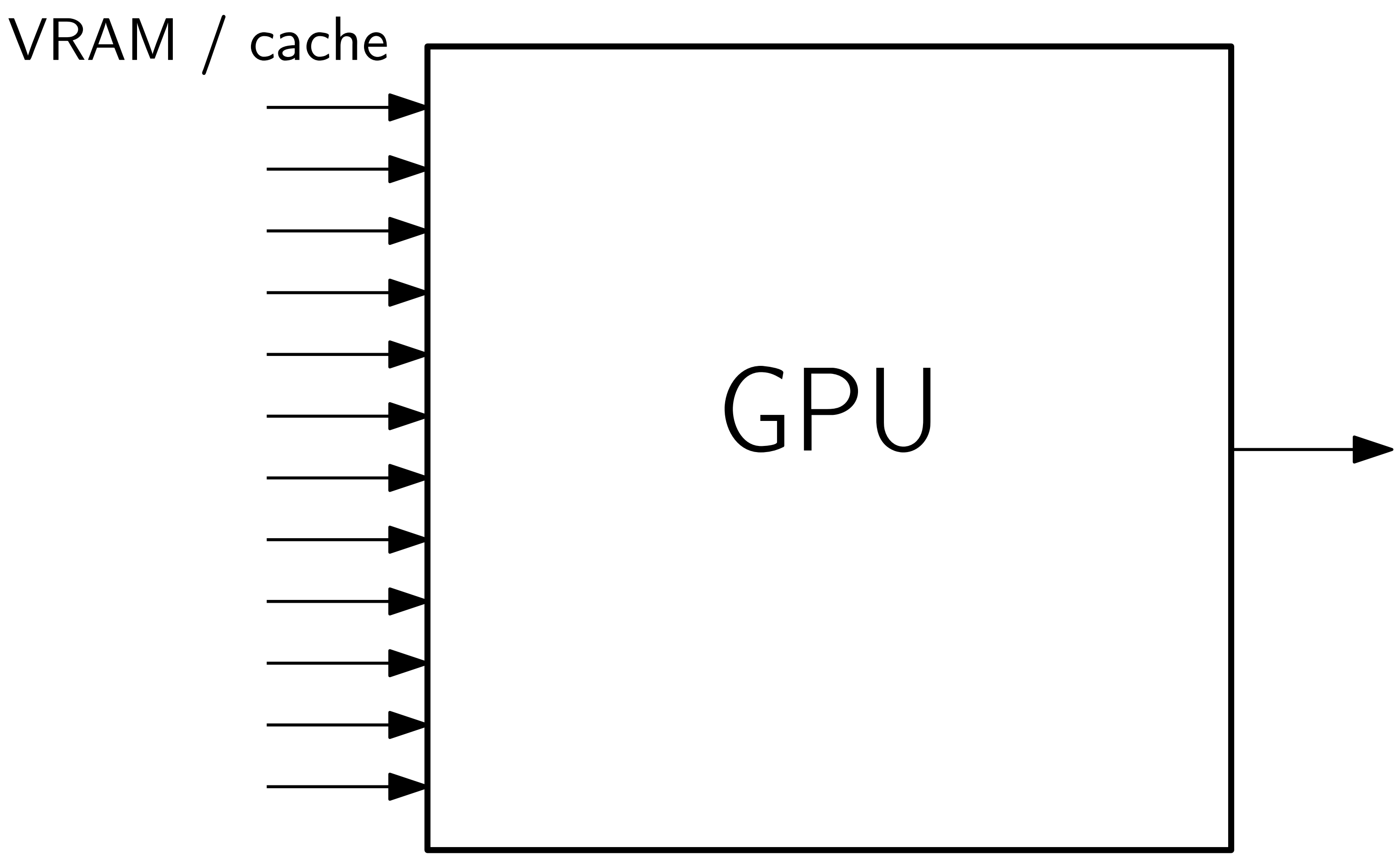

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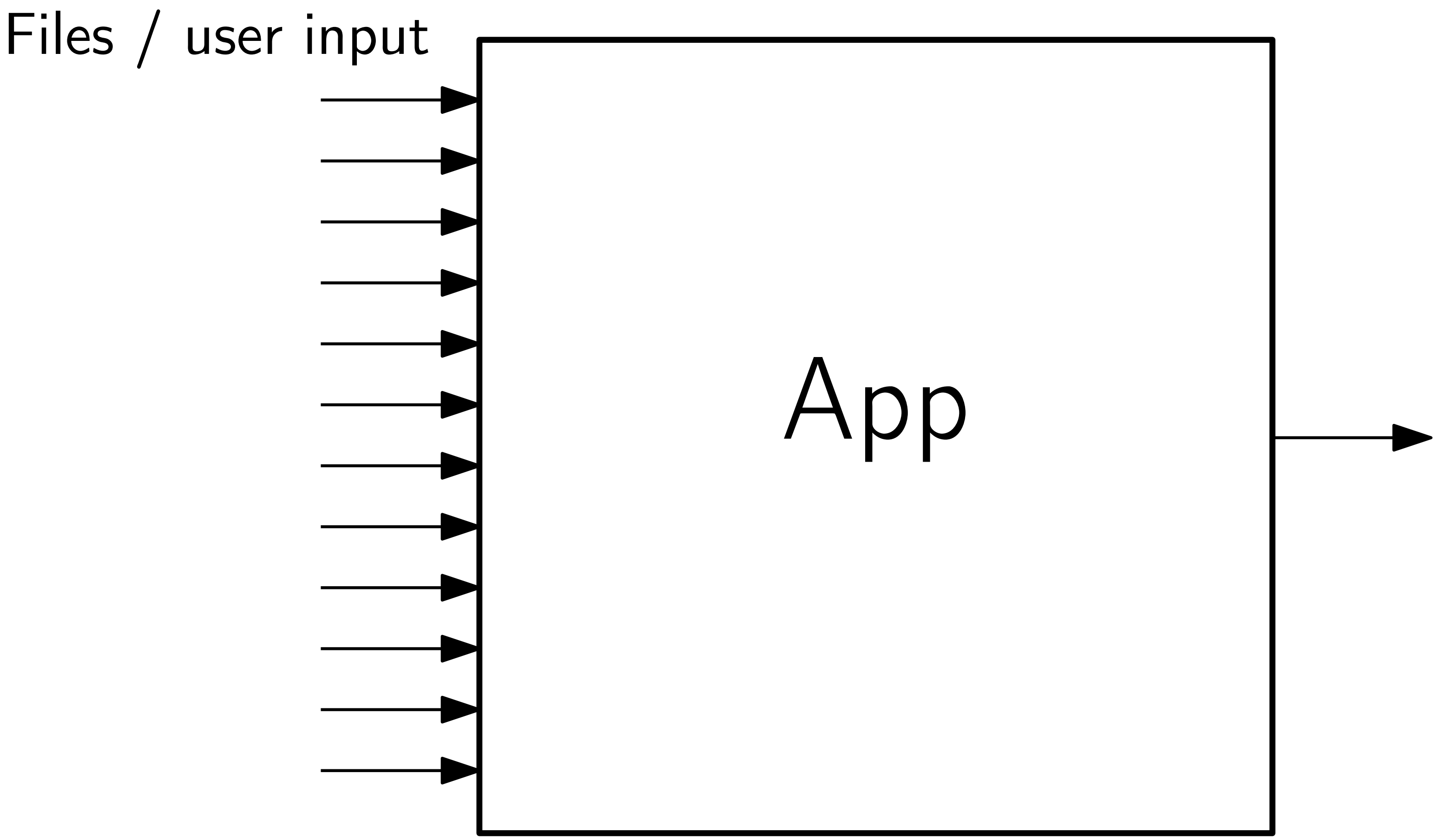

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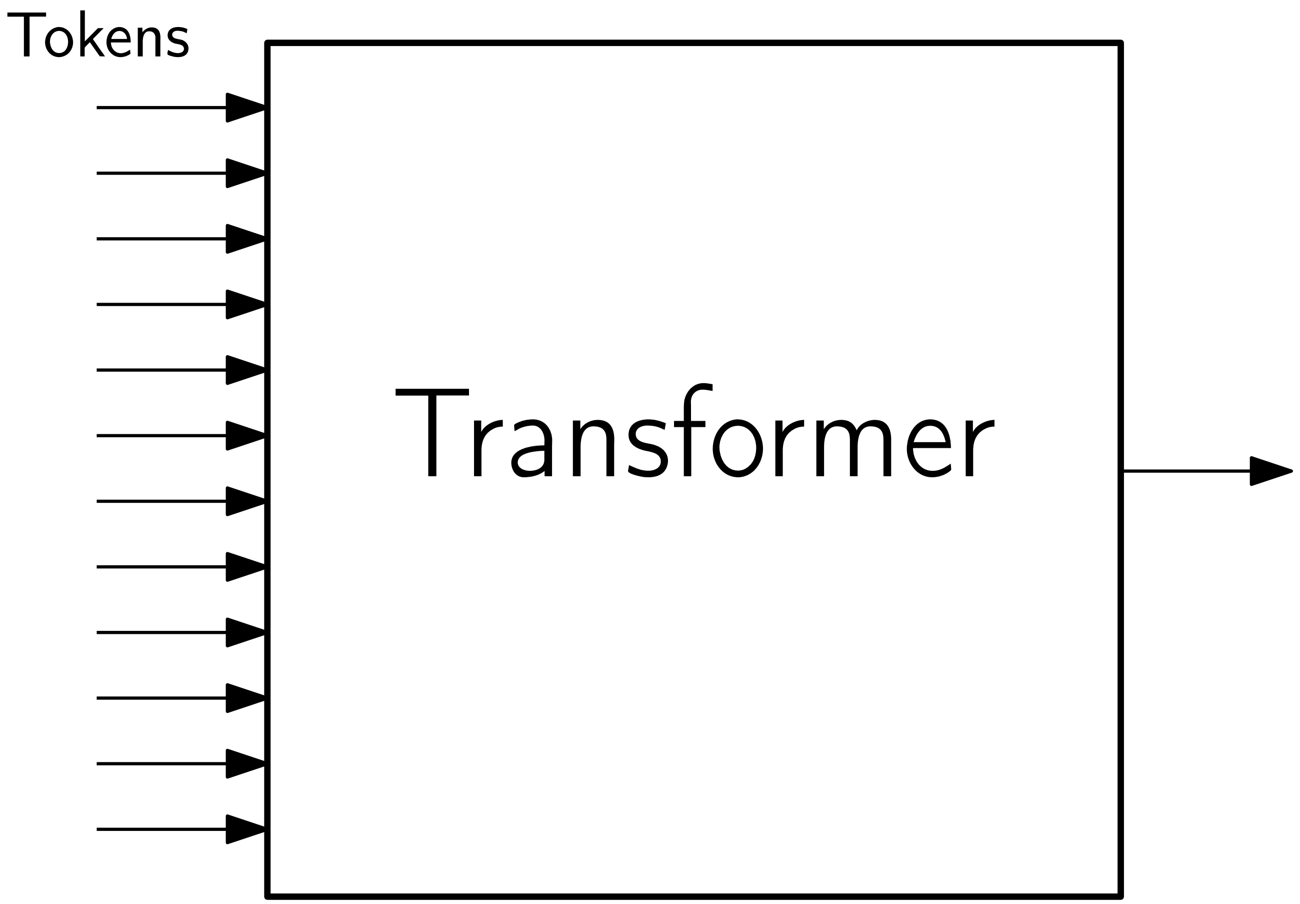

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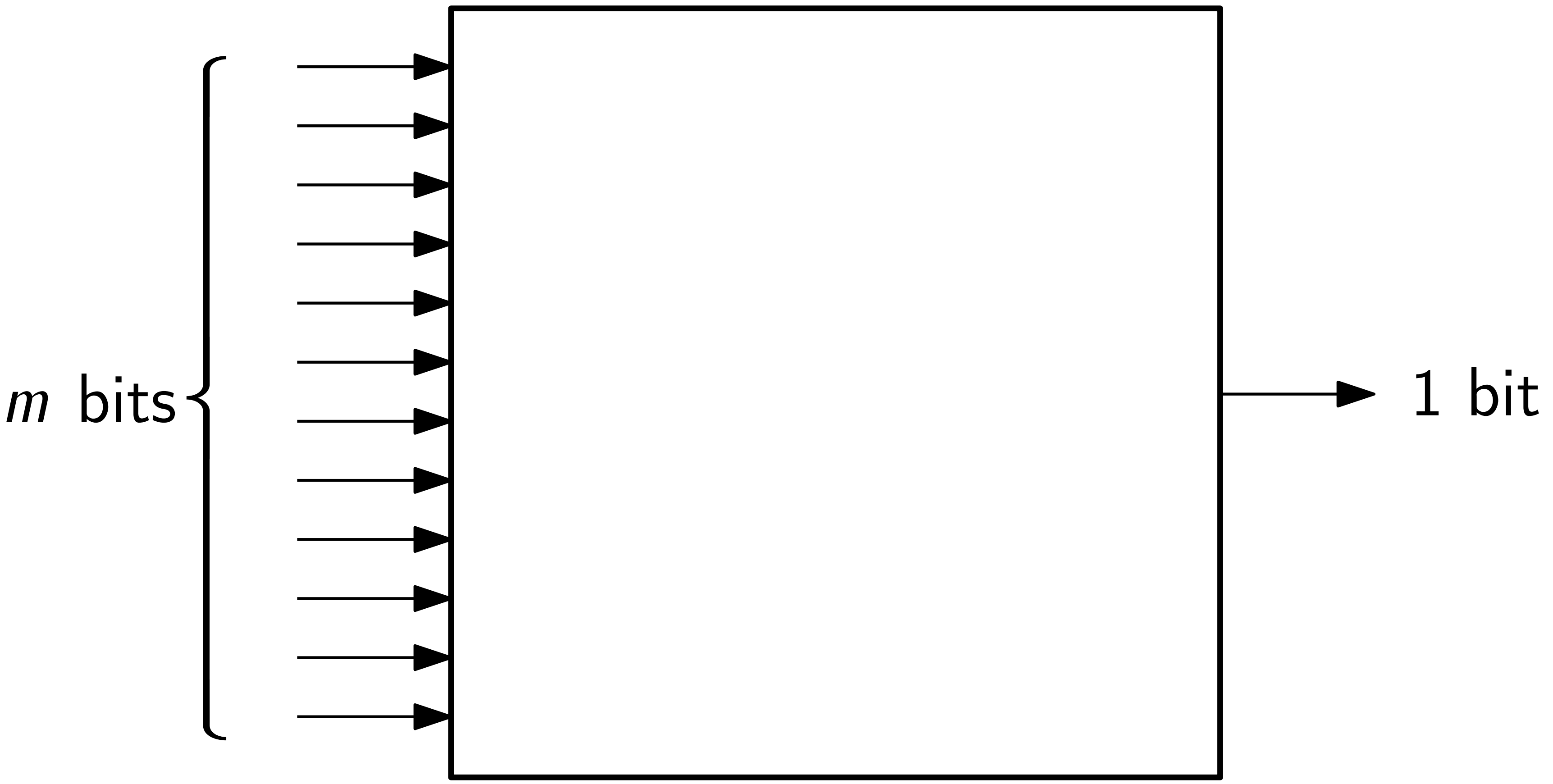

Function $\{0,1\}^m \to \{0,1\}$

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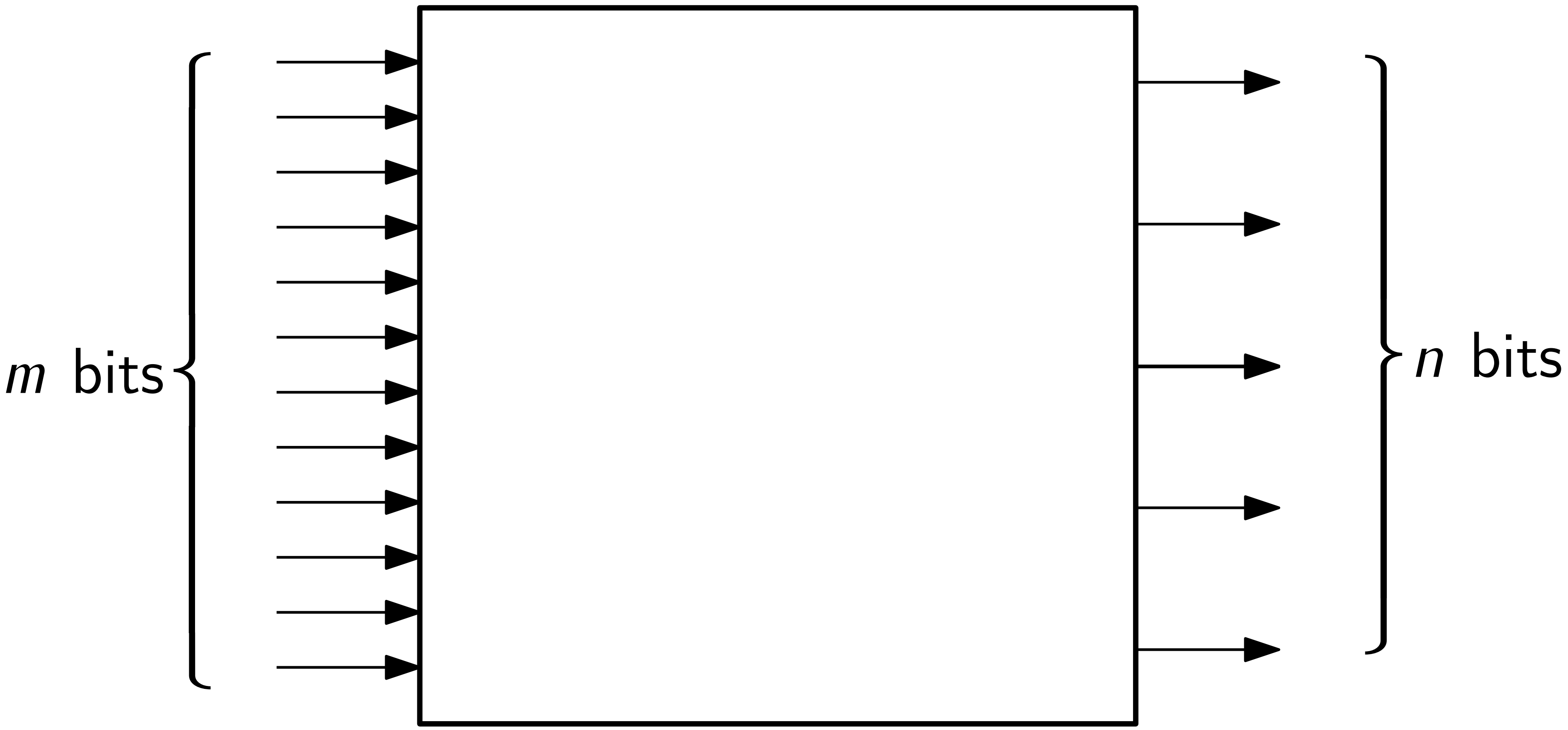

Function $\{0,1\}^m \to \{0,1\}^n$

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Concrete application: modelize the behavior of any digital system using logic/electronic circuits

p_1	p_2	p_3	p_4	p_5	p_6	?
0	0	0	0	0	0	$out_{(0,0,0,0,0,0)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b_1	b_2	b_3	b_4	b_5	b_6	$out_{(b_1,b_2,b_3,b_4,b_5,b_6)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	1	1	1	1	$out_{(1,1,1,1,1,1)}$

Procedure:

- Make a list L of all satisfying assignments $(b_1, \ldots, b_n)$
- $L \subseteq \{0, 1\}^n$
- For each $(b_1, \ldots, b_n) \in L$, define $\psi_{(b_1,\ldots,b_n)}$ by

$$\bigwedge_{i\in\{1,\ldots,n\}:b_i=1} p_i \wedge \bigwedge_{i\in\{1,\ldots,n\}:b_i=0} \neg p_i$$

- φ is the formula

$$\bigvee_{(b_1,\ldots,b_n)\in L} \psi_{(b_1,\ldots,b_n)}$$

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Concrete application: modelize the behavior of any digital system using logic/electronic circuits

p_1	p_2	p_3	?	
0	0	0	0	
0	0	1	1	$(\neg p_1 \wedge \neg p_2 \wedge p_3)$
0	1	0	1	$(\neg p_1 \wedge p_2 \wedge \neg p_3)$
0	1	1	0	
1	0	0	1	$(p_1 \wedge \neg p_2 \wedge \neg p_3)$
1	0	1	0	
1	1	0	0	
1	1	1	1	$(p_1 \wedge p_2 \wedge p_3)$

Procedure:

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p_1	p_2	p_3	φ
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Procedure:

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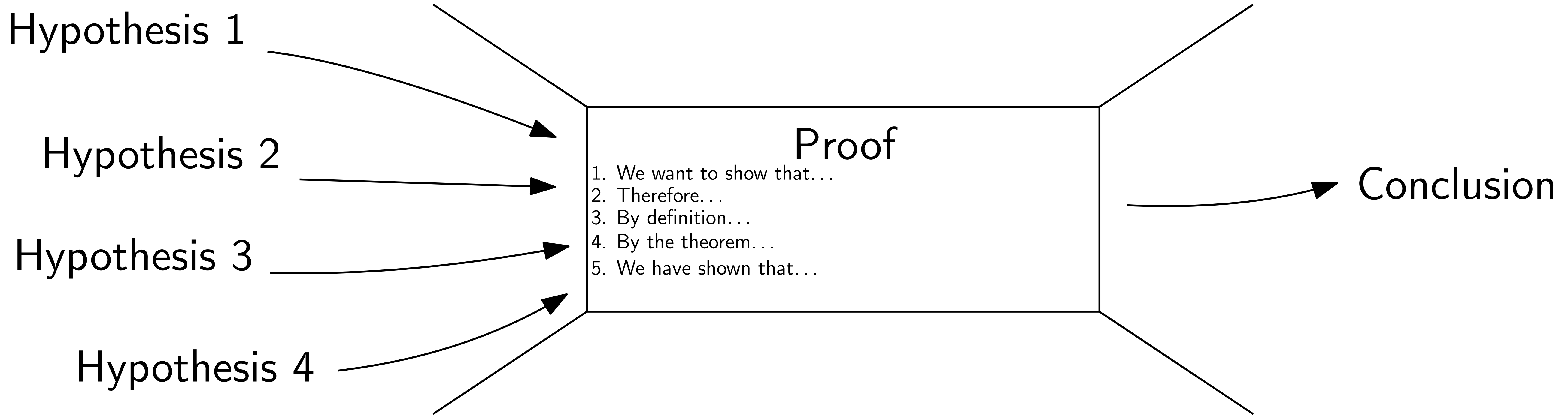
- φ is the formula

$$\bigvee_{(b_1,\ldots,b_n) \in L} \psi_{(b_1,\ldots,b_n)}$$

$\varphi : (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3)$

Proof methods



- A proof is a sequence of steps, like a cooking recipe. At each step:
  - Apply a definition
  - Apply a hypothesis
  - Apply a logical rule

How to prove that $\varphi \Rightarrow \psi$ is true?

Direct proof

1. Assume that φ is true.
2. <insert rest of the proof>
3. We obtain that ψ is true.

Proof by contraposition

1. Assume that ψ is false.
2. <insert rest of the proof>
3. We obtain that φ is false.

Proof by contradiction

How to prove that $\varphi \Rightarrow \psi$ is true?

Proof by contraposition
1. Assume that ψ is false.
2. <insert rest of the proof>
3. We obtain that φ is false.

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	$\neg q$	$\neg p$	$(\neg q) \Rightarrow (\neg p)$
0	0	1	1	1
0	1	0	1	1
1	0	1	0	0
1	1	0	0	1

$$(p \Rightarrow q) \equiv ((\neg q) \Rightarrow (\neg p))$$

$$(\varphi \Rightarrow \psi) \equiv ((\neg \psi) \Rightarrow (\neg \varphi))$$

How to prove that $\varphi \Rightarrow \psi$ is true?

Proof by contradiction

1. Assume that ϕ is true and that ψ is false.
2. <insert rest of the proof>
3. We obtain something we know is false.

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	$\neg q$	$p \wedge \neg q$	$\perp$	$(p \wedge \neg q) \Rightarrow \perp$
0	0	1	0	0	1
0	1	0	0	0	1
1	0	1	1	0	0
1	1	0	0	0	1

$$(p \Rightarrow q) \equiv ((p \wedge \neg q) \Rightarrow \perp)$$

$$(\varphi \Rightarrow \psi) \equiv ((\varphi \wedge \neg \psi) \Rightarrow \perp)$$

"Best" proof method: maximizes the number of hypotheses!

Predicate Logic

- Want: a logic in which we can express statements like

Some animal flies *Every mammal does not fly* *For every x, if $x > 10$ then $2x > 20$*

quantifiers predicates

Definition. The formulas of predicate logic are built as follows:

- Every predicate with variables $x, y, z, \ldots$ is a formula
- If φ and ψ are predicate formulas, then $\varphi \wedge \psi, \varphi \vee \psi, \varphi \Rightarrow \psi, \neg\varphi$ are also formulas
- If φ is a formula, then $\forall x \varphi$ is a formula
- If φ is a formula, then $\exists x \varphi$ is a formula

"x is a mammal", "$x > y$", "$x \in A$"

x is a mammal $\Rightarrow \neg(x$ flies$)$

$x > 10 \Rightarrow 2x > 20$

$\forall x(x$ is a mammal $\Rightarrow \neg x$ flies$)$

$\exists x(x$ is an animal $\wedge\ x$ flies$)$

$\forall x$: For all possible values of x, ...

$\exists x$: There exists a possible value of x such that...

$\forall x \in S$: for all x in the set S, ...

$\exists x \in S$: there exists some x in the set S such that...

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- Because of proofs by contradiction/contraposition, one often has to compute $\neg\varphi$ for a formula φ
- Here are some good shortcuts to remember for negation:

If φ is...	Then $\neg\varphi$ is equivalent to...
$\psi \wedge \theta$	$\neg\psi \vee \neg\theta$
$\psi \vee \theta$	$\neg\psi \wedge \neg\theta$
$\neg\psi$	ψ
$\psi \Rightarrow \theta$	$\psi \wedge \neg\theta$
$\forall x\, \psi$	$\exists x\, \neg\psi$
$\exists x\, \psi$	$\forall x\, \neg\psi$

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$\psi \vee \theta$	$\neg\psi \wedge \neg\theta$
$\neg\psi$	ψ
$\psi \Rightarrow \theta$	$\psi \wedge \neg\theta$
$\forall x \in S\, \psi$	$\exists x \in S\, \neg\psi$
$\exists x \in S\, \psi$	$\forall x \in S\, \neg\psi$

Theorem. Let $a \in \mathbb{R}$. If $a^2 = 0$, then $a = 0$.

$$\forall \ a \in \mathbb{R} \quad (a^2 = 0 \implies a = 0)$$

Proof.

1. By contradiction, suppose that $a^2 = 0$ and $a \neq 0$.
2. Since $a \neq 0$, $\frac{1}{a}$ exists.
3. Then $a^2 \times \frac{1}{a} = 0 \times \frac{1}{a}$.
4. So $a = 0$.
5. Contradiction! $\square$

Theorem. Let $a, b \in \mathbb{N}$. If $a \times b$ is not divisible by 5, then a is not divisible by 5.

$$\forall a \in \mathbb{N} \forall b \in \mathbb{N} \quad (\neg(a \times b \text{ divisible by } 5) \implies \neg(a \text{ is divisible by } 5))$$

Contraposition of the implication:

$$\forall a \in \mathbb{N} \forall b \in \mathbb{N} \quad (a \text{ is divisible by } 5 \implies a \times b \text{ divisible by } 5)$$

Proof.

1. By definition of being divisible by 5, we can write $a = 5 \times a'$ for some $a' \in \mathbb{N}$.
2. Then $a \times b = 5 \times a' \times b$.
3. So $a \times b$ is divisible by 5. $\square$