Automata Theory and Formal Languages

Summer Term 2026

3rd Lecture

Grammars

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Grammars - Motivating example

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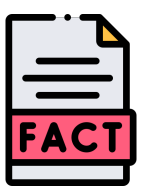

Observations.

- *article* **the** is followed by the *noun* **students**
- both together are a *nominal expression*
- **study** is a *verb*
- **automata theory** is also a nominal expression

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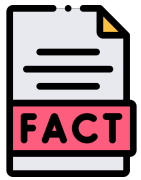

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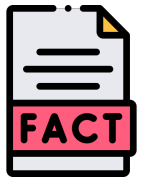

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sentence $\Rightarrow$ *nominal expression verb nominal expression*

$\Rightarrow$ *article noun verb nominal expression*

$\Rightarrow$ The students *verb nominal expression*

$\Rightarrow$ The students study *nominal expression*

$\Rightarrow$ The students study automata theory

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$G = (\{S\}, \{a\}, \mathcal{P}, S)$ with $\mathcal{P} = \{S \rightarrow Sa, S \rightarrow a\}$

Generating languages from grammars

A grammar $G = (V, \Sigma, \mathcal{P}, S)$ *generates* a language $L(G)$ by applying the production rules from $\mathcal{P}$ successively, starting from the start variable S until no variables are present anymore.

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The derivation process.

A string $u \in (V \cup \Sigma)^*$ can be transformed into a string $v \in (V \cup \Sigma)^*$ if

$$u = xyz$$
$$v = xy'z \quad \text{with} \quad x, z \in (V \cup \Sigma)^*$$

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Short: $u \underset{G}{\Rightarrow} v$ or $u \Rightarrow v$ if G can be determined from context.

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For a grammar G, the language generated by G is given by

$$L(G) := \{w \in \Sigma^* \mid S \underset{G}{\Rightarrow}^* w\} \ .$$

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The generated language is given by:

$$\begin{aligned} L(G) &= \{w \in \{a\}^* \mid S \underset{G}{\Rightarrow}^* w\} \\ &= \{w \in \{a\}^* \mid S \Rightarrow Sa \Rightarrow \cdots \Rightarrow \underbrace{a \ldots a}_{n \text{ times}}, n \in \mathbb{N}\} \\ &= \{a^n \mid n \in \mathbb{N}\} \end{aligned}$$

Generating languages from grammars

$$G = (\{E, T, F\}, \{a, +, *\}, \mathcal{P}, E)$$

$$\mathcal{P} = \{E \to T, E \to E + T, T \to F,$$
$$T \to T * F, F \to a\}$$

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$$\mathcal{P} = \{E \to T, E \to E + T, T \to F,$$
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Then $a + a \in L(G)$ since:

$$E \Rightarrow E + T \Rightarrow T + T \Rightarrow F + T \Rightarrow a + T \Rightarrow a + F \Rightarrow a + a$$

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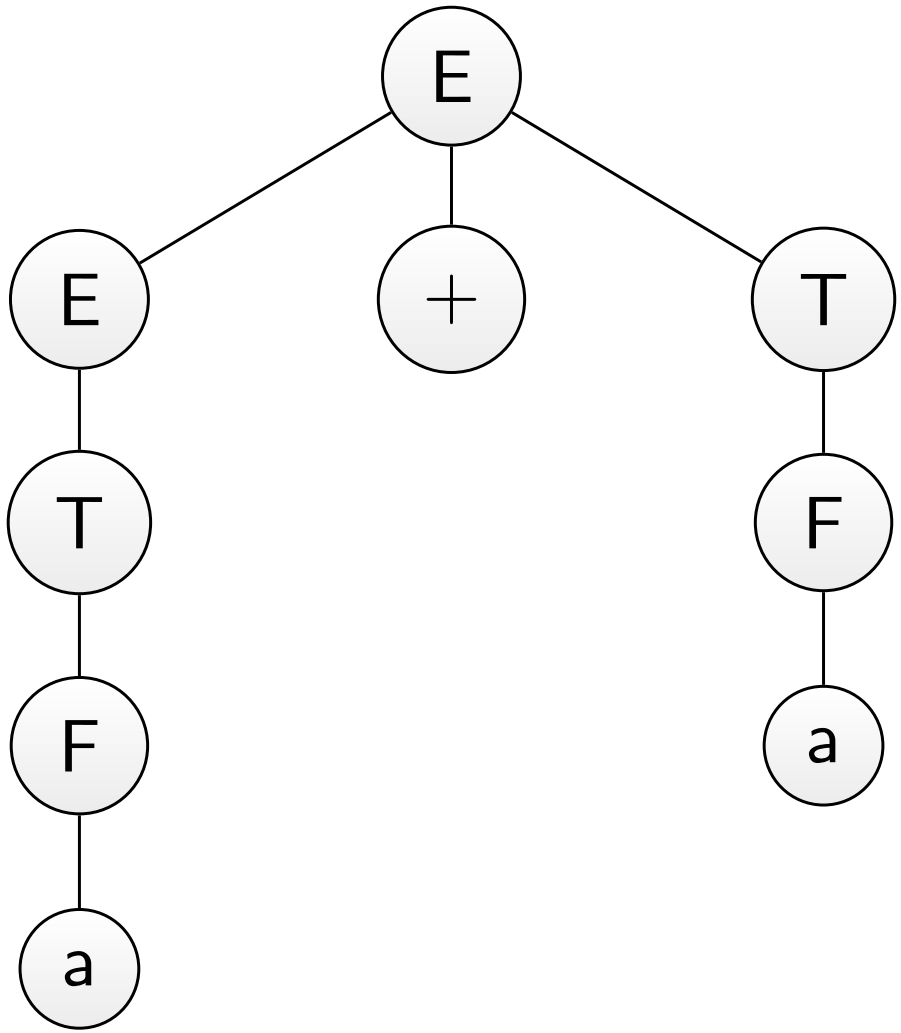
  - Add nodes $X_1, X_2, \dots X_k$ to tree.
  - connect $B - X_1, B - X_2, \dots, B - X_k$ with an edge

An example of a syntax tree

$$E \Rightarrow E + T \Rightarrow T + T \Rightarrow F + T \Rightarrow a + T \Rightarrow a + F \Rightarrow a + a$$



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- Derivation is not necessarily a deterministic, but generally a *non-deterministic* process.
- A derivation is only complete if the final string w contains only terminal symbols, i.e. $w \in \Sigma^*$.