Automata Theory and Formal Languages

Sommer Term 2026

5th Lecture

Regular Languages and Finite Automata

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Regular Languages and Automata

Regular Languages

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- type 3 grammars
- finite automata

Finite Automata

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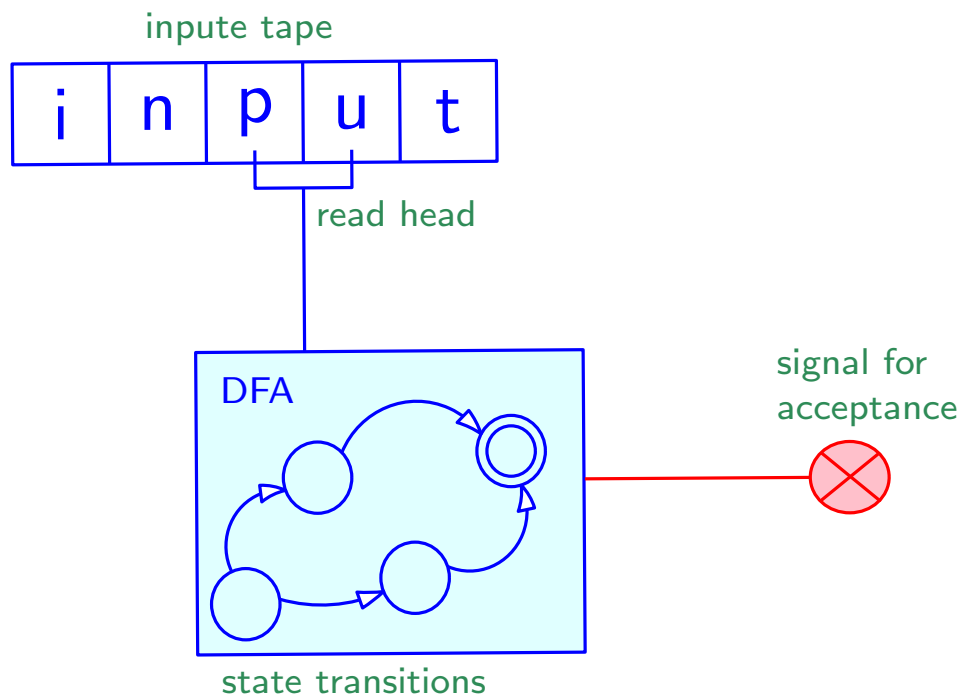
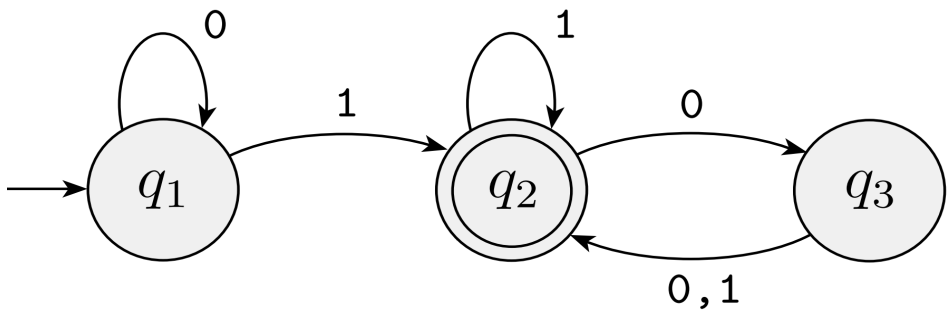


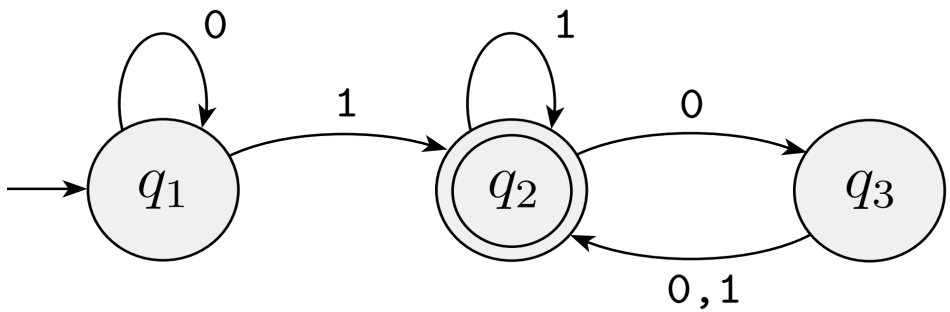
Finite Automata

Example



Finite Automata

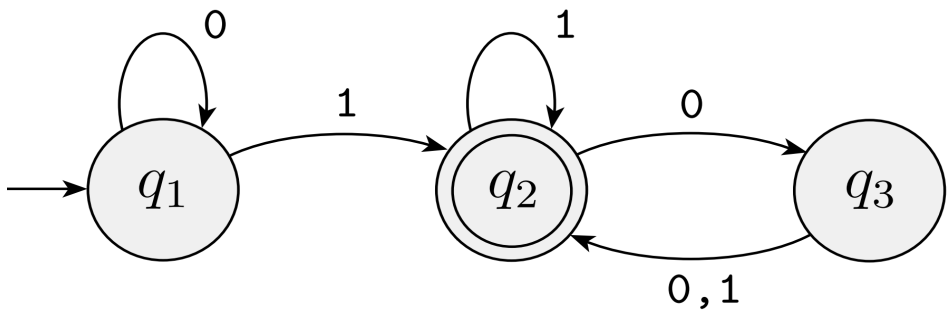
Example



On input 0100:

Finite Automata

Example

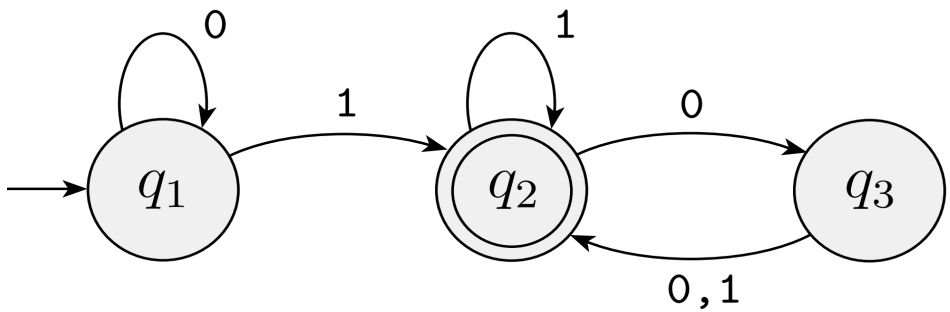


On input 0100:

q_1

Finite Automata

Example



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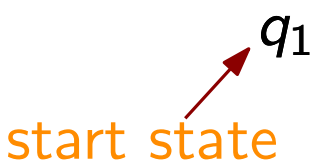
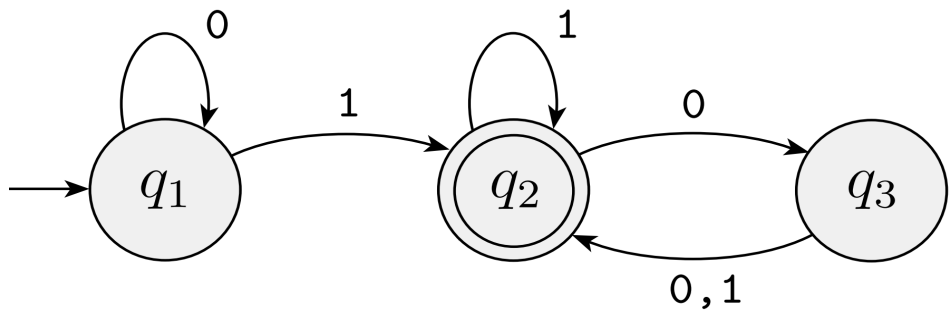


Finite Automata

Example



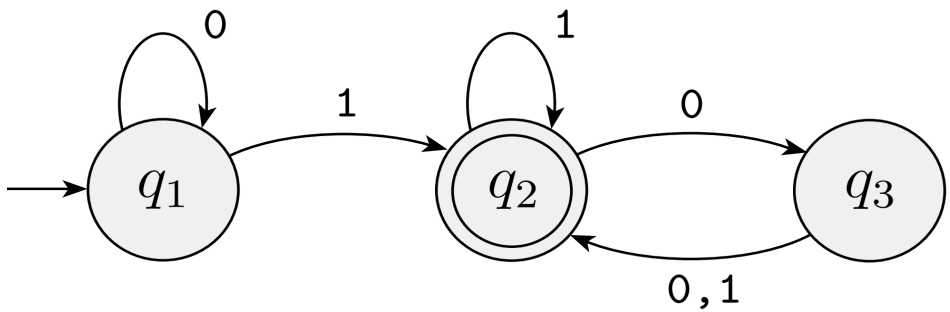
On input 0100:

$$q_1 \xrightarrow{0} q_1$$

start state

Finite Automata

Example



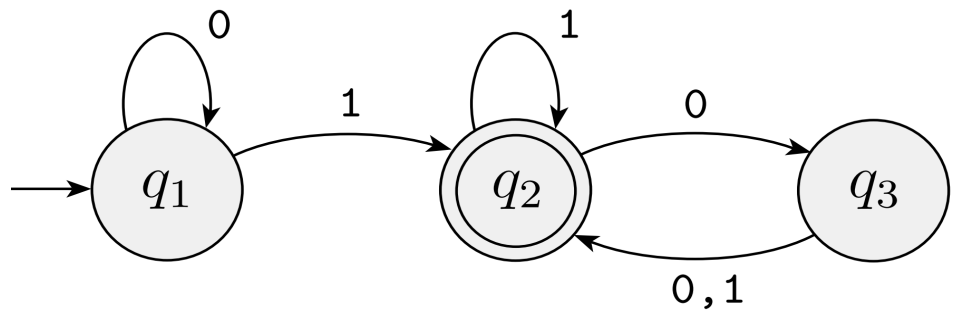
On input 0100:

$$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2$$

start state

Finite Automata

Example



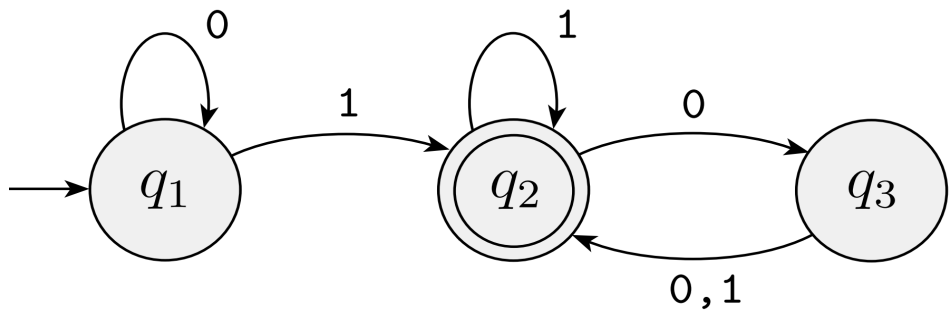
On input 0100:

$$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{0} q_3$$

start state

Finite Automata

Example



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Finite Automata

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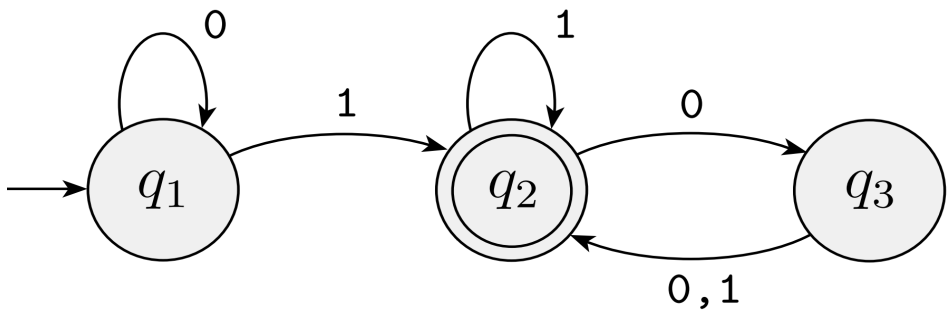

On input 0100:

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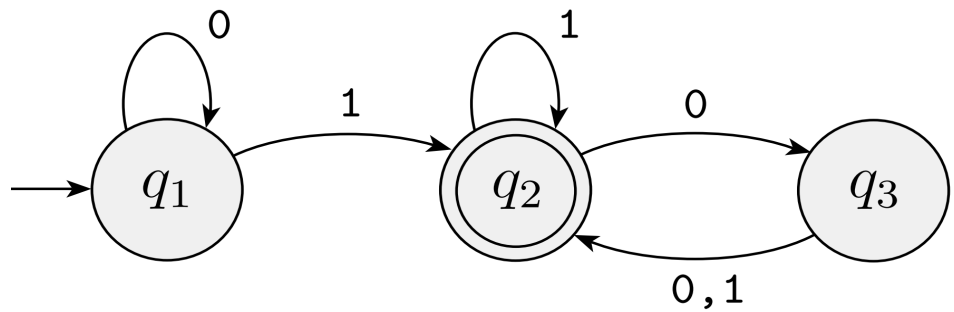
start state

accepting state

Finite Automata

Example



On input 0100:

$$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{0} q_3 \xrightarrow{0} q_2$$

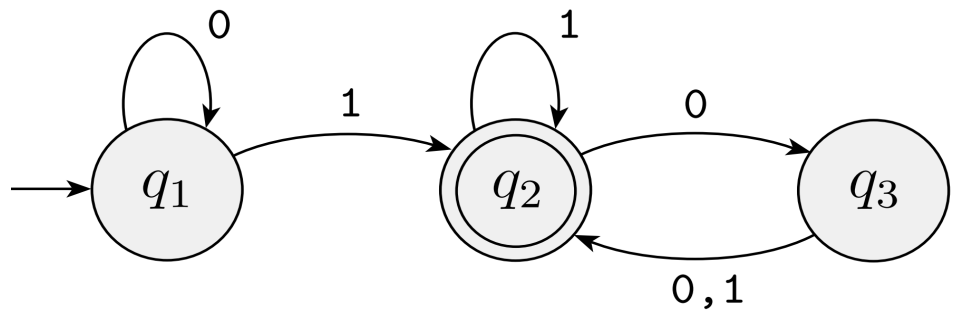
start state

accepting state

0100 is accepted.

Finite Automata

Example



On input 0100:

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start state

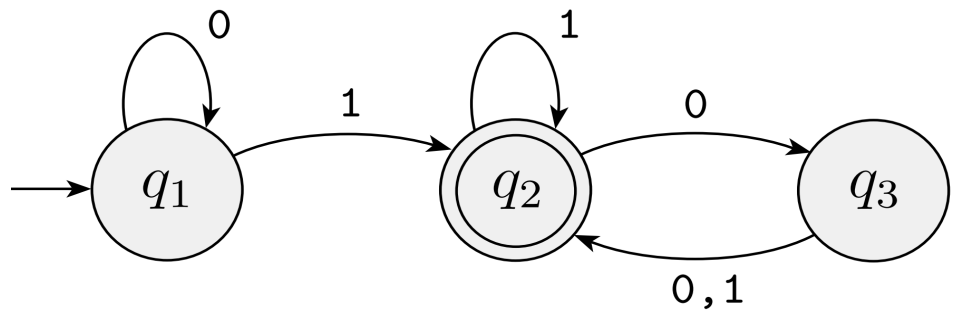
accepting state

0100 is accepted.

On input 0110:

Finite Automata

Example



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start state

accepting state

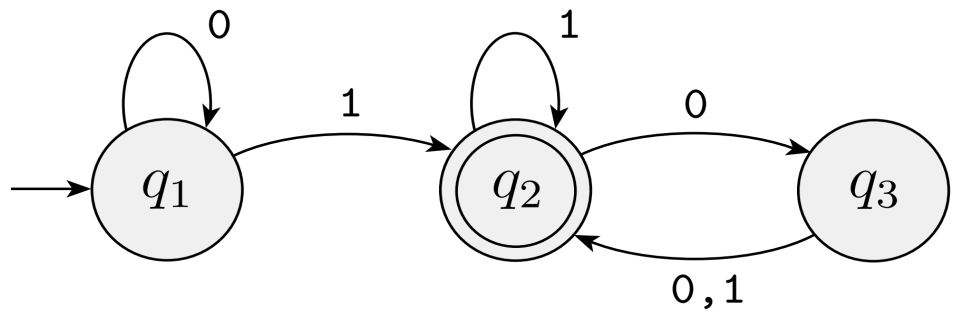
0100 is accepted.

On input 0110:

$$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_2 \xrightarrow{0} q_3$$

Finite Automata

Example



On input 0100:

$$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{0} q_3 \xrightarrow{0} q_2$$

start state

accepting state

0100 is accepted.

On input 0110:

non-accepting state

$$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_2 \xrightarrow{0} q_3$$

Finite Automata

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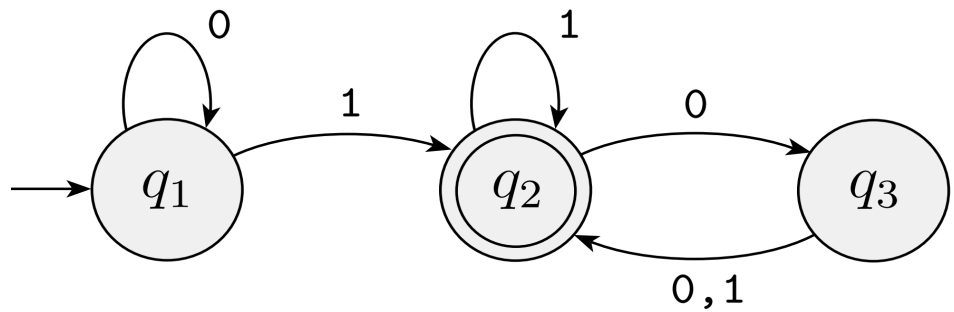

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start state

accepting state

0100 is accepted.

On input 0110:

non-accepting state

$$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_2 \xrightarrow{0} q_3$$

0110 is rejected.

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Definition

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$$M = (Z, \Sigma, \delta, z_0, E)$$

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- δ: $Z \times \Sigma \to Z$: State transition function

Graphical Representation

State diagram

directed graph with nodes representing the states and labeled edges representing the state transitions

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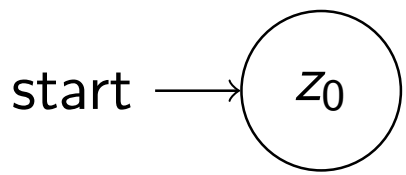
Initial state

labeled with an arrow



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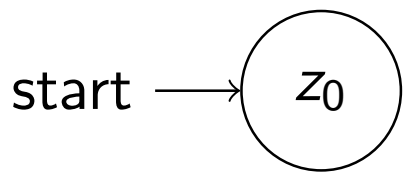
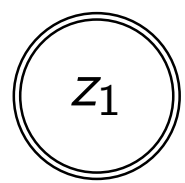


Final state

double circle



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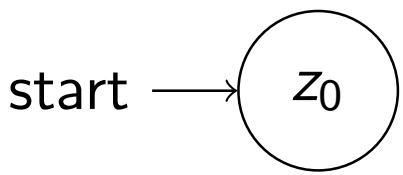

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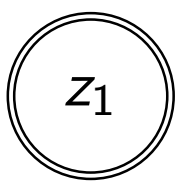
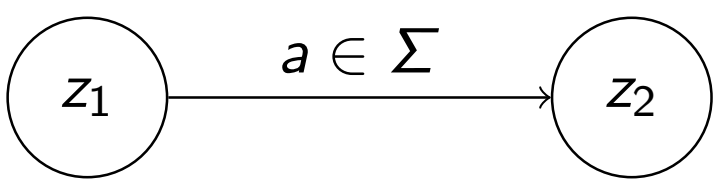
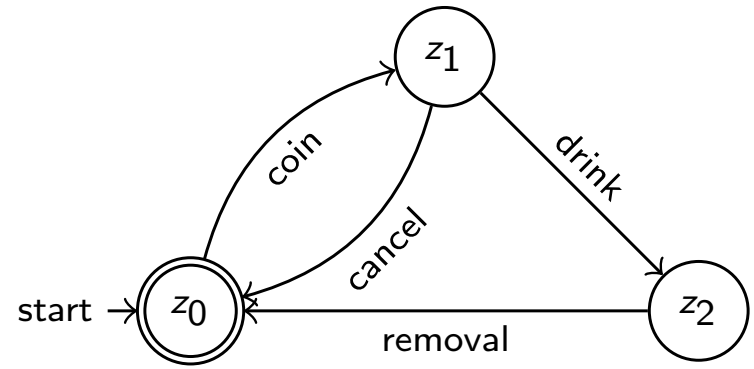


State transition

if $\delta(z_1, a) = z_2$



Example: Soda Machine



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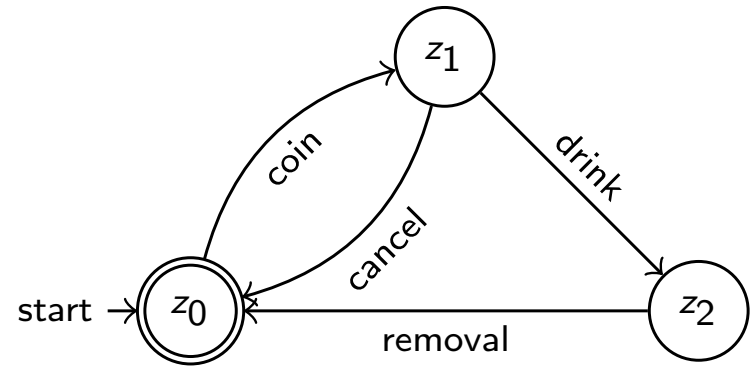
States: $Z = \{z_0, z_1, z_2\}$

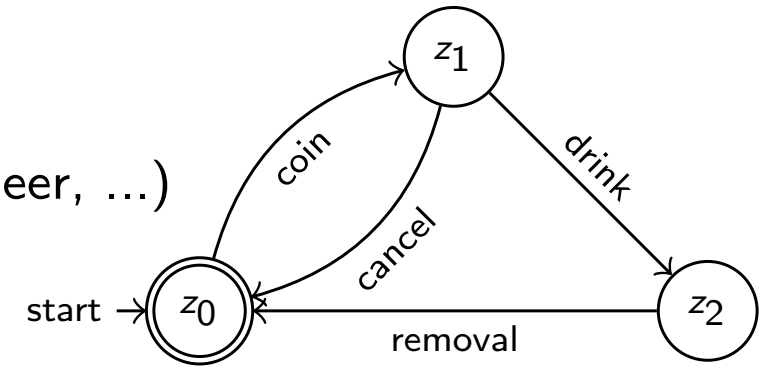


Example: Soda Machine

States: $Z = \{z_0, z_1, z_2\}$

z_0: machine waits for coins
z_1: machine waits for drink choice (Cola, Fanta, Beer, ...)
z_2: machine waits for drink removal

Initial state: z_0
Final state: z_0



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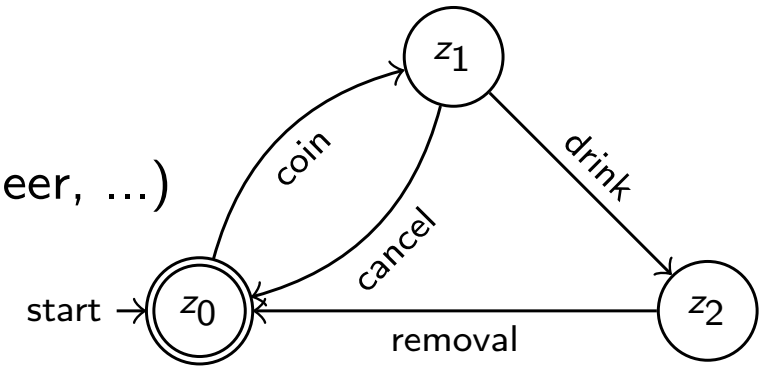

Input alphabet: $\Sigma = \{\text{coin}, \text{drink}, \text{cancel}, \text{removal}\}$

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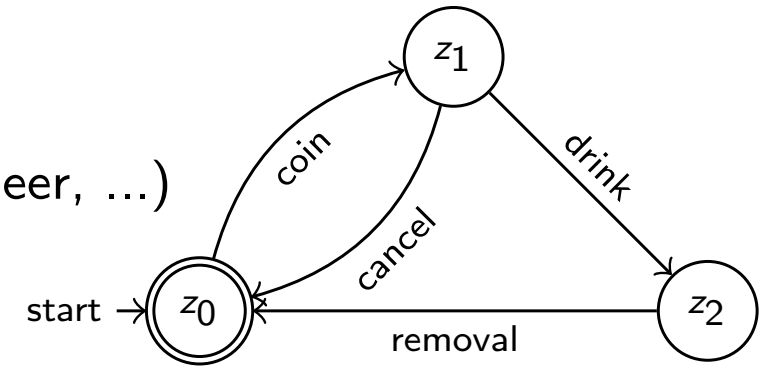
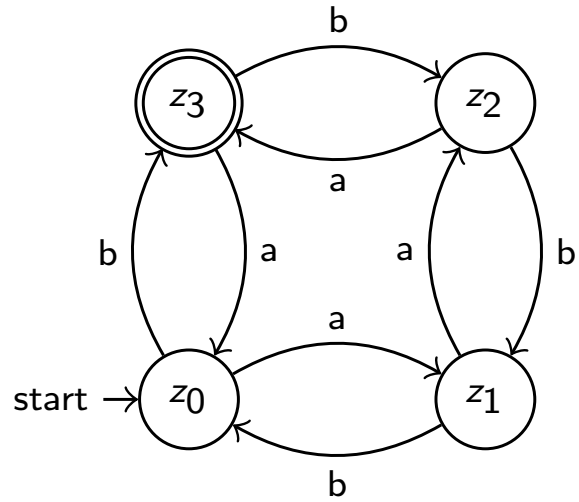


Input alphabet: $\Sigma = \{\text{coin}, \text{drink}, \text{cancel}, \text{removal}\}$

State transitions

$$\delta(z_0, \text{coin}) = z_1$$
$$\delta(z_1, \text{drink}) = z_2$$
$$\delta(z_2, \text{removal}) = z_0$$
$$\delta(z_1, \text{cancel}) = z_0$$

Further Example

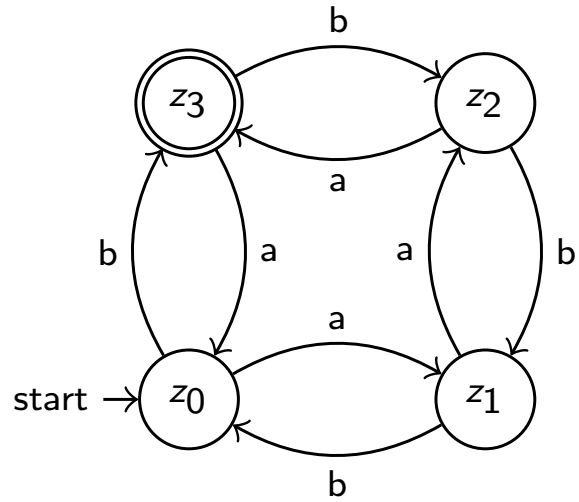


Further Example

States: $Z = \{z_0, z_1, z_2, z_3\}$

Initial state: z_0

Final state: z_3



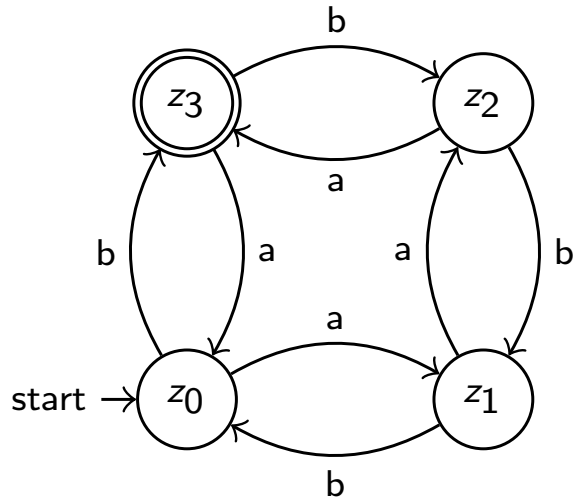
Further Example

States: $Z = \{z_0, z_1, z_2, z_3\}$

Initial state: z_0

Final state: z_3

Input alphabet: $\Sigma = \{a, b\}$



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State transitions

$\delta(z_0, a) = z_1,\ \delta(z_0, b) = z_3,\ \delta(z_1, a) = z_2$
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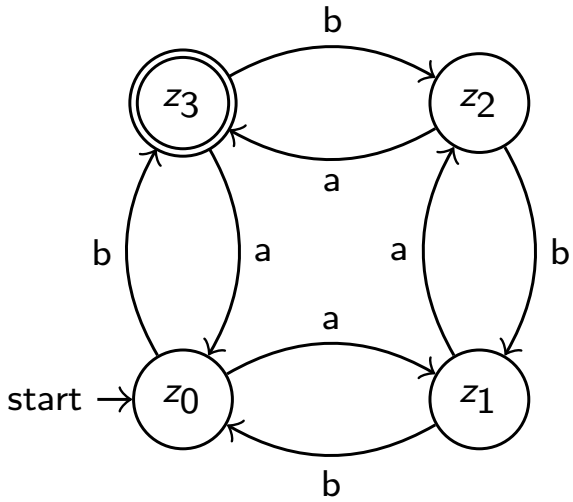

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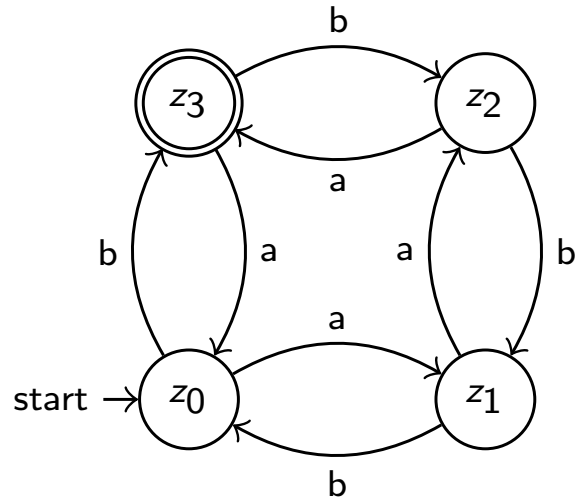

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State transitions - Table representation

δ	a	b
z_0	z_1	z_3
z_1	z_2	z_0
z_2	z_3	z_1
z_3	z_0	z_2

Language Accepted by a DFA

Definition

Let $M = (Z, \Sigma, \delta, z_0, E)$ be a deterministic finite automaton. Then we define the function $\hat{\delta} : Z \times \Sigma^* \to Z$ inductively:

$$\hat{\delta}(z, \varepsilon) = z$$

$$\hat{\delta}(z, ax) = \hat{\delta}(\delta(z, a), x)$$

for $a \in \Sigma$, $x \in \Sigma^*$, $z \in Z$.

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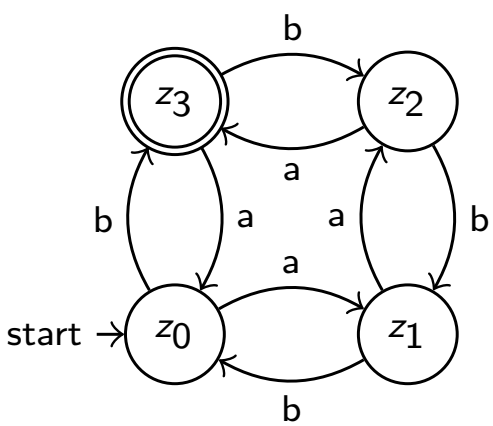

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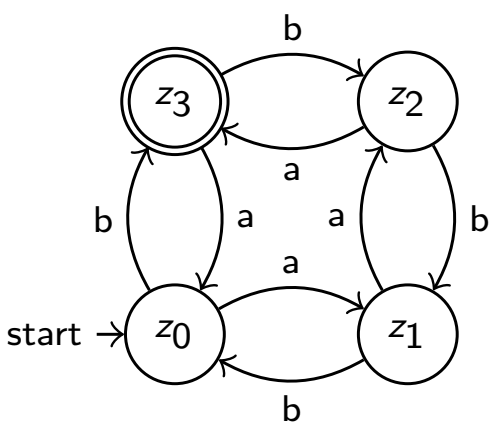

$$\hat{\delta}(z_0, aab) = z_1$$
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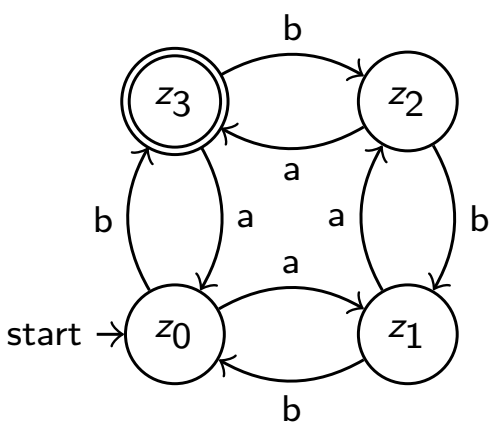

$$\hat{\delta}(z_0, aab) = z_1$$
$$\hat{\delta}(z_2, bbbaaa) = z_2$$
$$\hat{\delta}(z_i, b^n a^n) = z_i$$

Language Accepted by a DFA

Language of a DFA

The language accepted or recognized by M is then given by

$$L(M) := \{x \in \Sigma^* \mid \hat{\delta}(z_0, x) \in E\}.$$

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- $\hat{\delta}$ extends δ from characters to words

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Remarks

- $\hat{\delta}$ extends δ from characters to words
- For characters we have $\hat{\delta}(z, a) = \delta(z, a)$
  (in this case we do not differentiate between length 1 words and characters)

Language Accepted by a DFA

Language of a DFA

The language accepted or recognized by M is then given by

$$L(M) := \{x \in \Sigma^* \mid \hat{\delta}(z_0, x) \in E\}.$$

Remarks

- $\hat{\delta}$ extends δ from characters to words
- For characters we have $\hat{\delta}(z, a) = \delta(z, a)$
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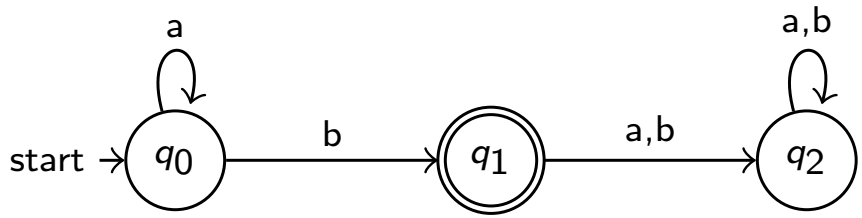
Language Accepted by a DFA

Example

Automaton M



Language Accepted by a DFA

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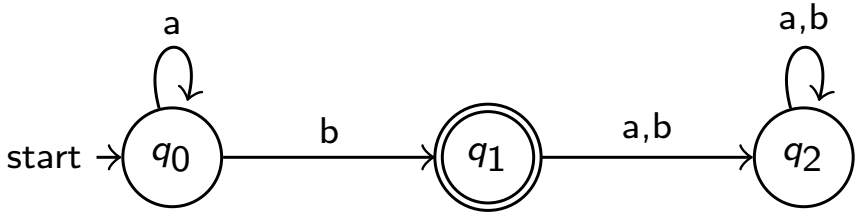

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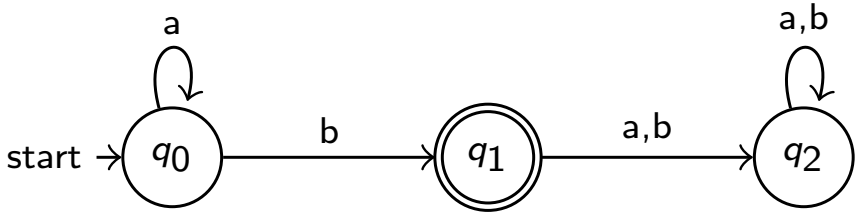

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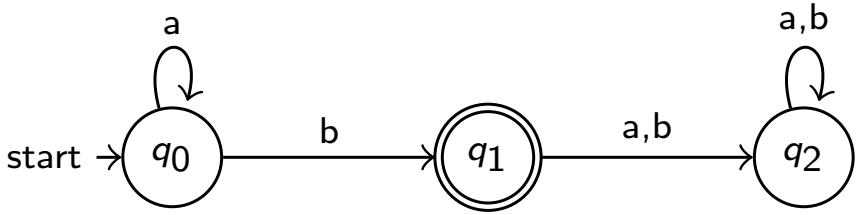

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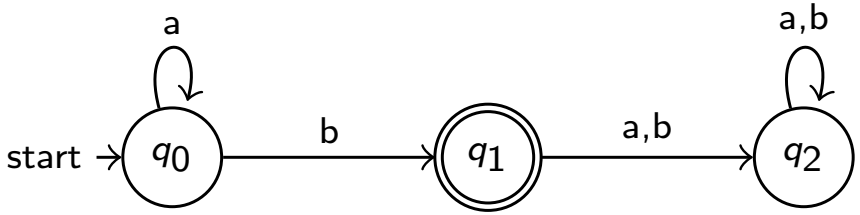

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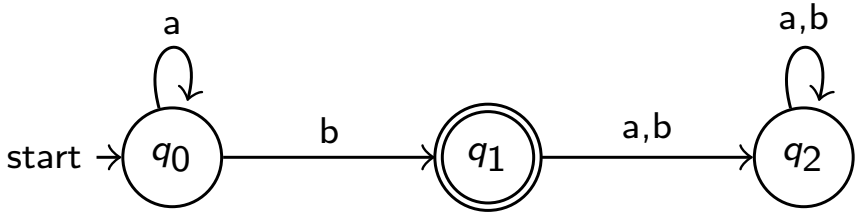

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Language Accepted by a DFA

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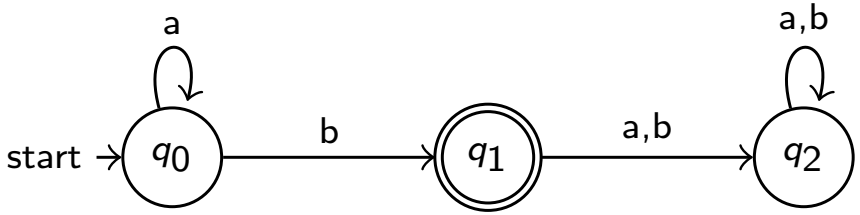

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Language Accepted by a DFA

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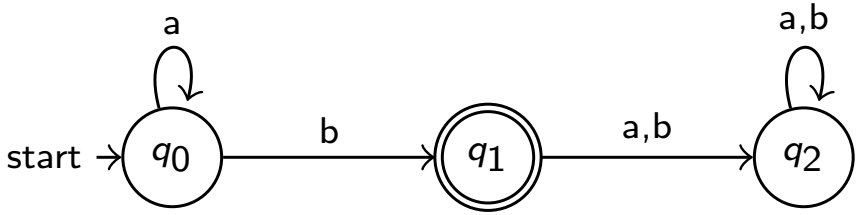

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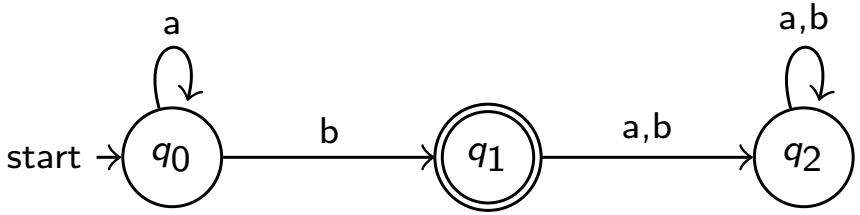

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Language Accepted by a DFA

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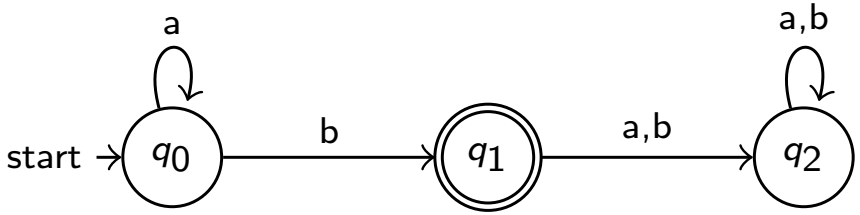

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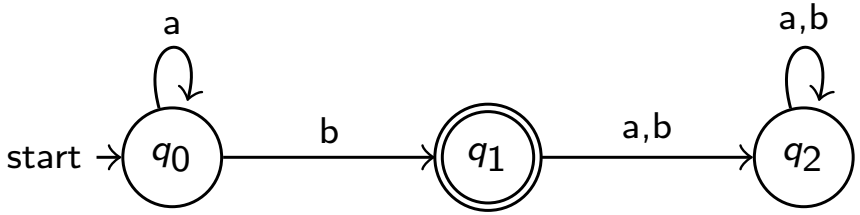

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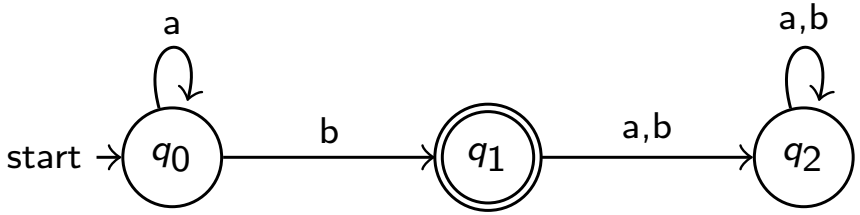

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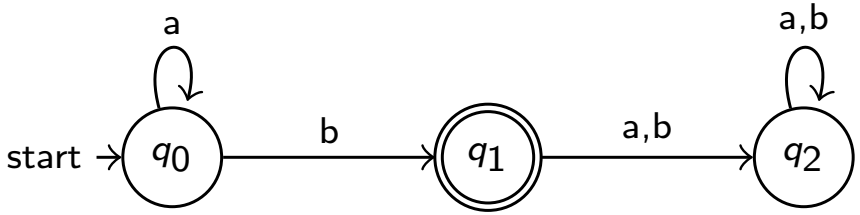

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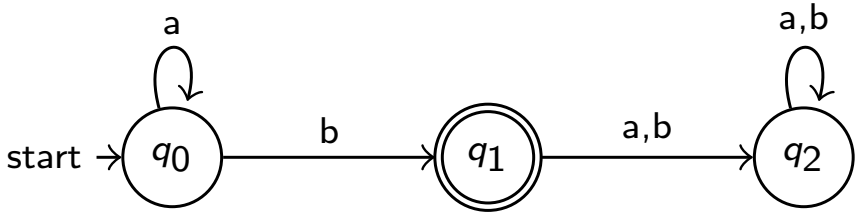

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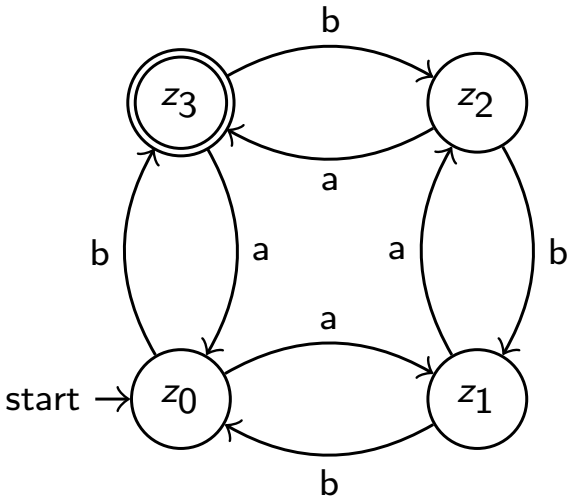
Further Example



Accepting language?

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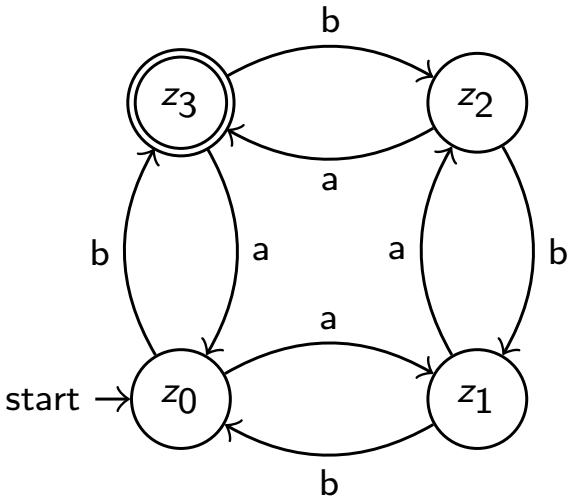

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$$L = \{x \in \{a, b\}^* \mid (\#a \text{ in } x) - (\#b \text{ in } x) \equiv 3 \pmod 4\}$$

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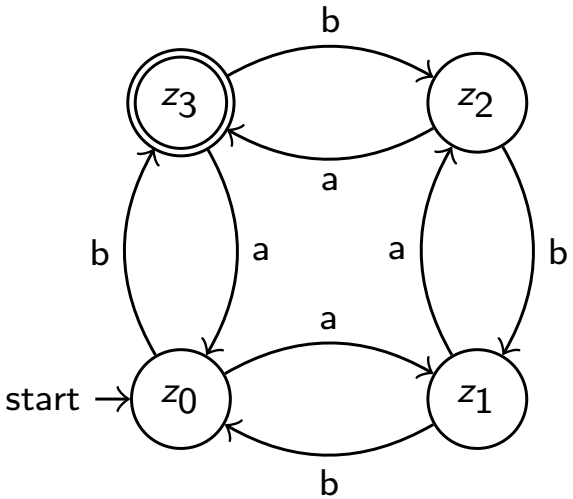

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Try to prove it formally.

Constructing DFA for a Language

Example

Construct an automaton M recognizing
$A = \{x \in \{\mathsf{a}, \mathsf{b}\}^* : |x|_{\mathsf{a}}$ is even and $|x|_{\mathsf{b}}$ is odd$\}$

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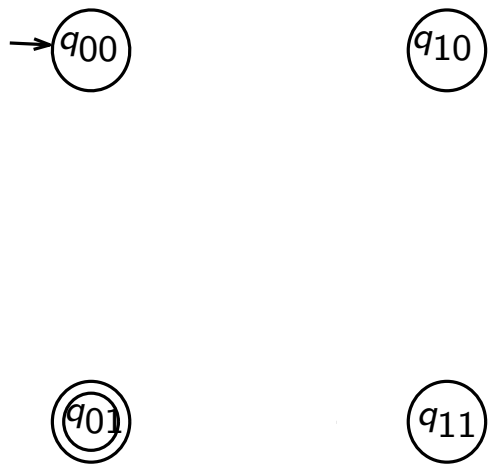

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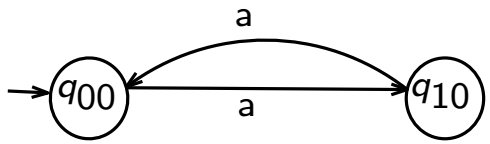

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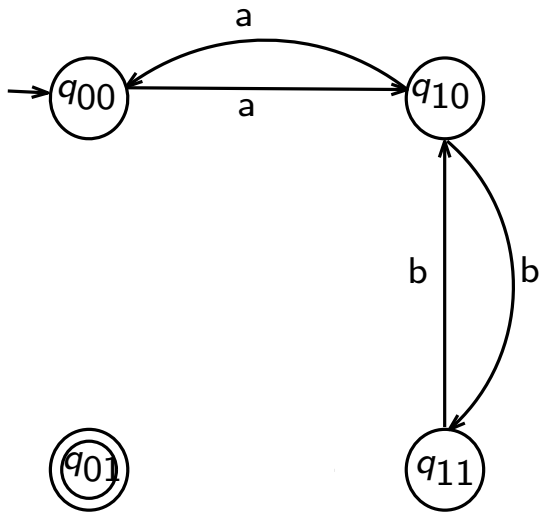

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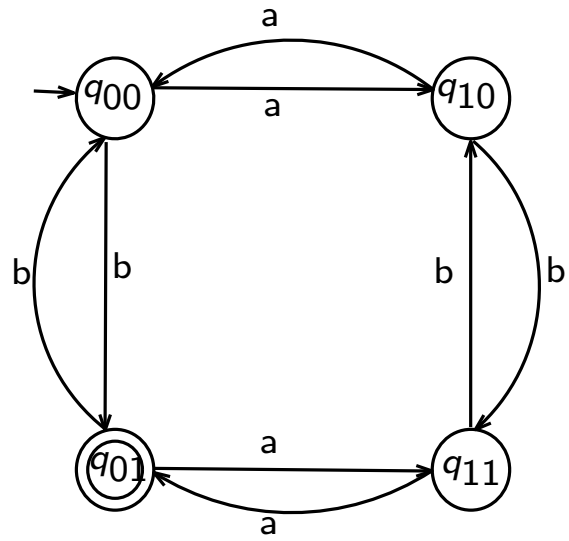

DFA: Regular Languages

Theorem

Every language that can be accepted by a DFA is generated by a regular (type 3) grammar.

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Grammar	Languages	Production rules (constraints)
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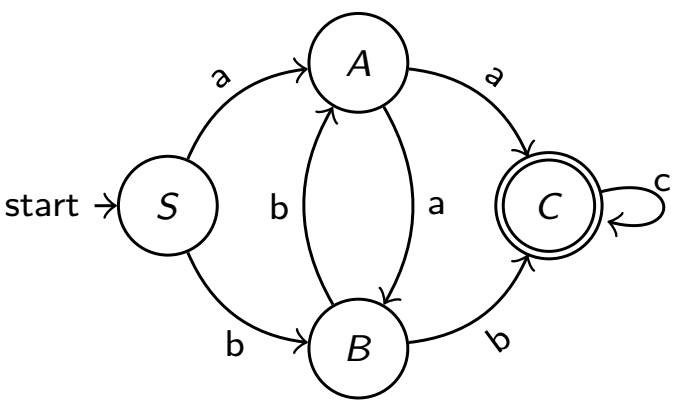

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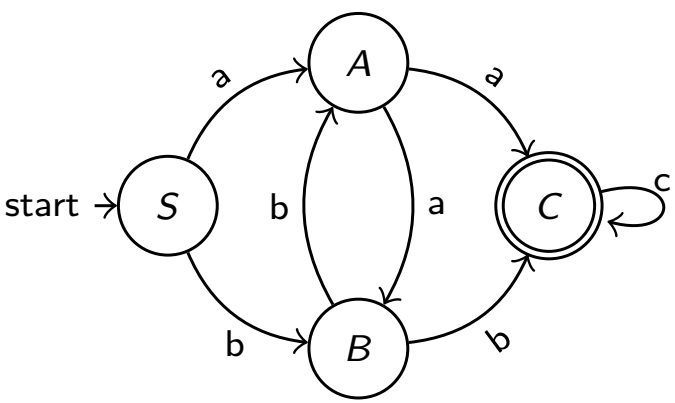

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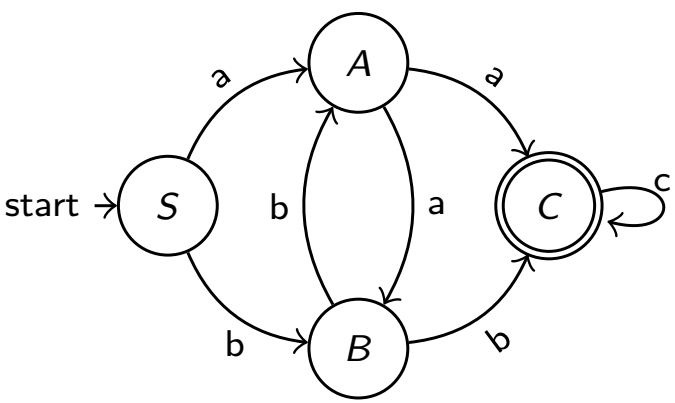

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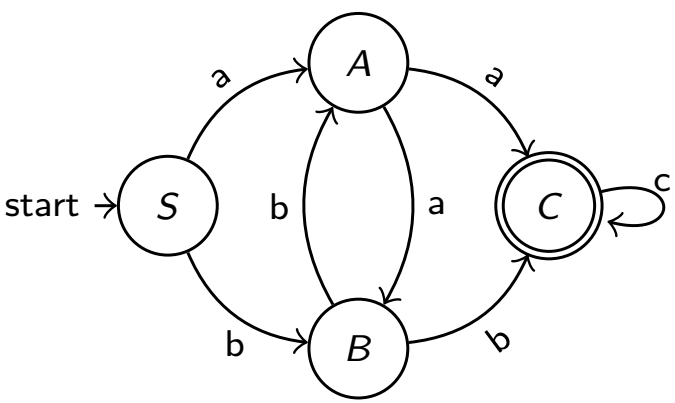

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$C \to \mathrm{c}C$

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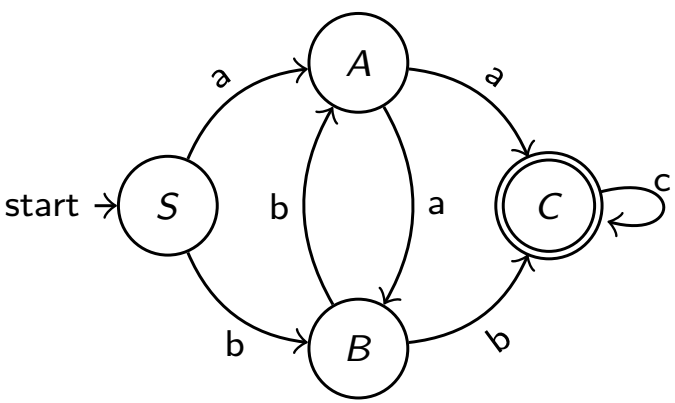

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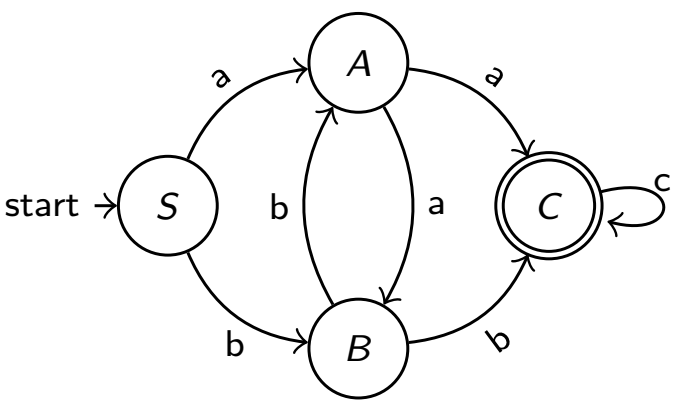

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- if $z_0 \in E$ then $(z_0 \to \varepsilon) \in P$
- for each transition $\delta(z_1, a) = z_2$, put the following rule:

$$(z_1 \to a z_2) \in P$$

DFA: Regular Languages

Proof of Theorem

Let $M = (Z, \Sigma, \delta, z_0, E)$ be an automaton.

We construct a regular grammar $G = (V, \Sigma_G, P, S)$ with

$$V = Z, \ \Sigma_G = \Sigma, \ S = z_0,$$

and rules P:

- if $z_0 \in E$ then $(z_0 \to \varepsilon) \in P$
- for each transition $\delta(z_1, a) = z_2$, put the following rule:

$$(z_1 \to a z_2) \in P$$

and if $z_2 \in E$:

$$(z_1 \to a) \in P$$

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Proof of Theorem (cont'd)

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$\Leftrightarrow$ $\exists$ sequence $z_0, z_1, \ldots, z_n$ of states of M with z_0 the initial state and $z_n \in E$ such that

$$\delta(z_0, a_1) = z_1,\ \delta(z_1, a_2) = z_2,\ \ldots,\ \delta(z_{n-1}, a_n) = z_n.$$

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$$z_0 \Rightarrow a_1 z_1 \Rightarrow a_1 a_2 z_2 \Rightarrow \cdots \Rightarrow a_1 a_2 \ldots a_{n-1} z_{n-1} \Rightarrow a_1 a_2 \ldots a_{n-1} a_r$$

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$\Leftrightarrow$ $x \in L(G)$

DFA: Total State Transition

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Mathematically one can associate a total function $f' : X \to Y \cup \{\bot\}$ for a partial function $f : X \to Y$ with:

$$f'(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{Def}(f) \\ \bot & \text{else} \end{cases}$$

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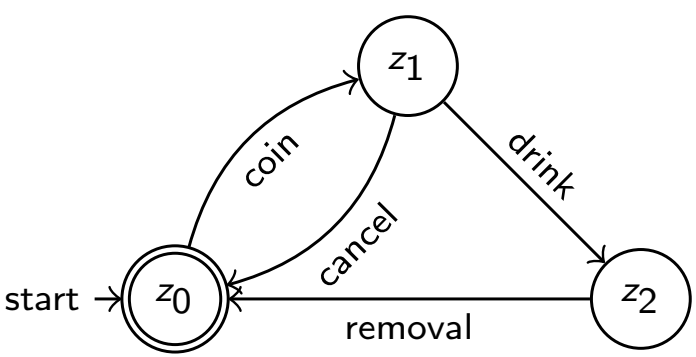
Example: Soda Machine



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