Automata Theory and Formal Languages

Summer Term 2025

6th Lecture

Nondeterministic Finite Automata

Prof. Dr. Matthias Mnich *Institute for Algorithms and Complexity*

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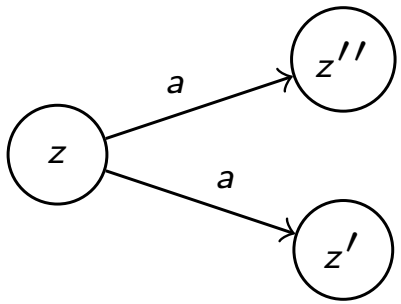

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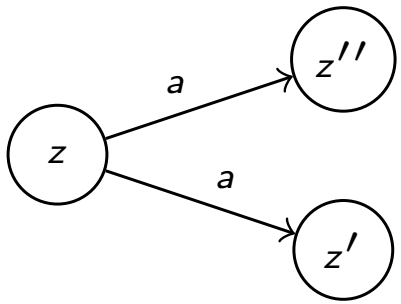

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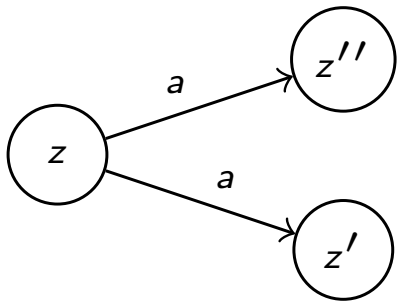

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- An NFA can have multiple initial states.

Example of non-deterministic finite automaton

An NFA over the alphabet $\{0, 1\}$

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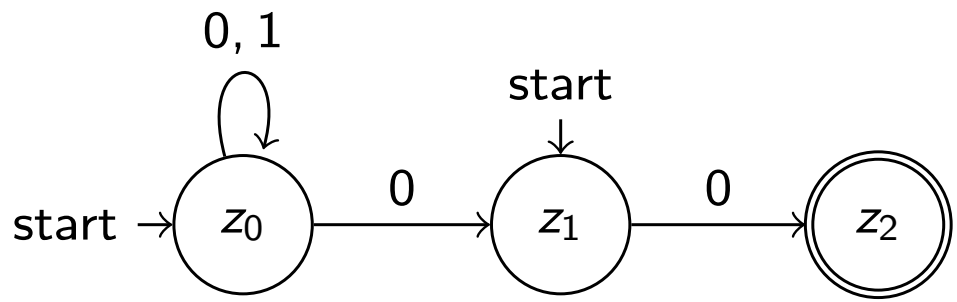

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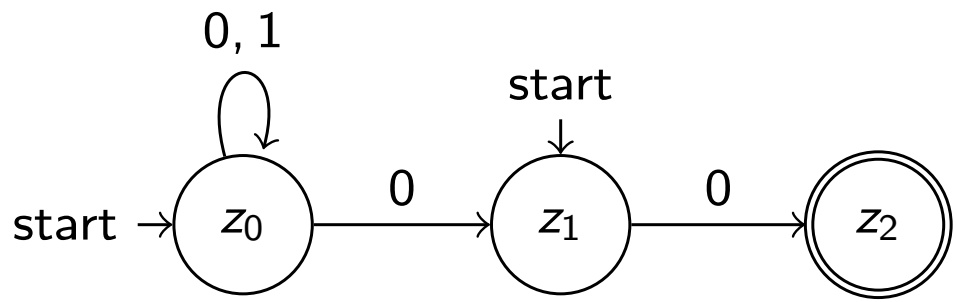

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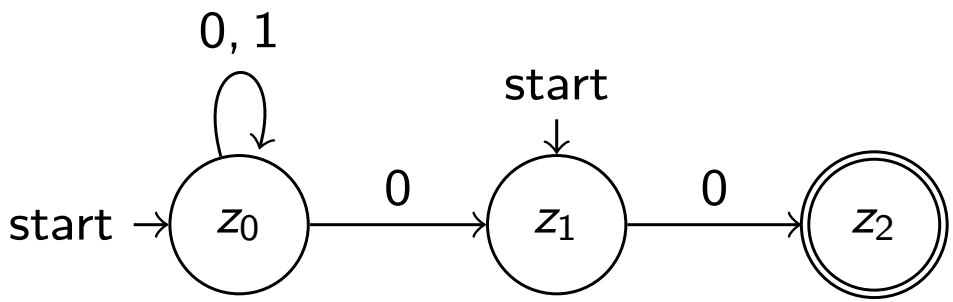

Which strings does this automaton accept?

It accepts strings ending with 00 or the length one string 0.

Recall: Power sets

For a set X, its *power set* is family

$$\mathcal{P}(X) := \{U \mid U \subsetneq X\}$$

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Remark

- $\emptyset \in \mathcal{P}(X)$
- $X \in \mathcal{P}(X)$
- The power set has $|\mathcal{P}(X)| = 2^{|X|}$ elements.

NFAs: Formal definition

An NFA M is a 5 tuple

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- δ: $Z \times \Sigma \to \mathcal{P}(Z)$: State transition function

Power Set Construction

Extended transition function $\hat{\delta}$

δ can again be generalized from characters to strings:

$$\hat{\delta} : \mathcal{P}(Z) \times \Sigma^* \to \mathcal{P}(Z)$$

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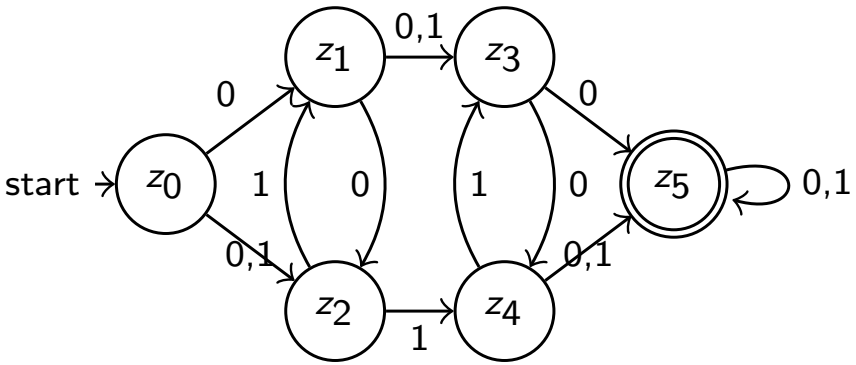

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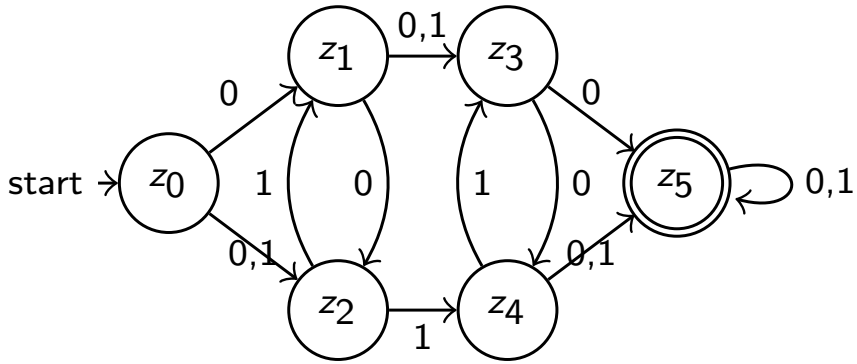

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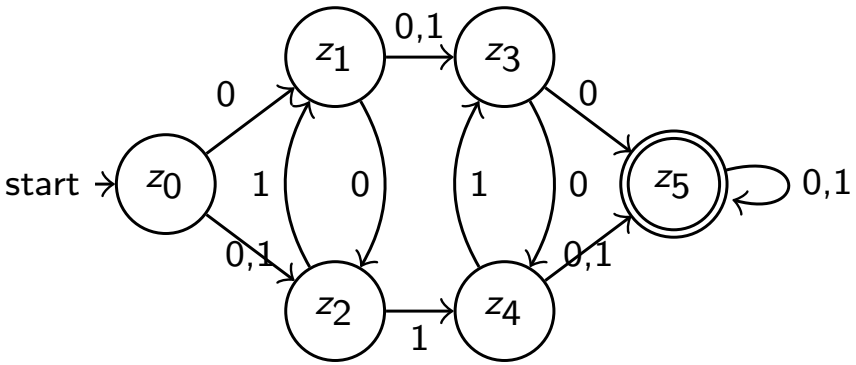

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$$\hat{\delta}(\{z_2, z_4\}, 110) = \{z_4, z_5\}$$

Equivalence of NFAs and DFAs

For a non-deterministic finite automaton $\mathcal{A} = (Z, \Sigma, \delta, S, E)$, the *language accepted by* $\mathcal{A}$ is

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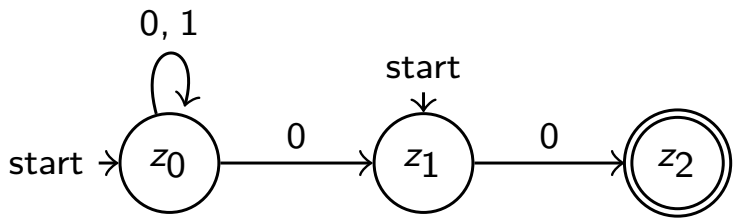

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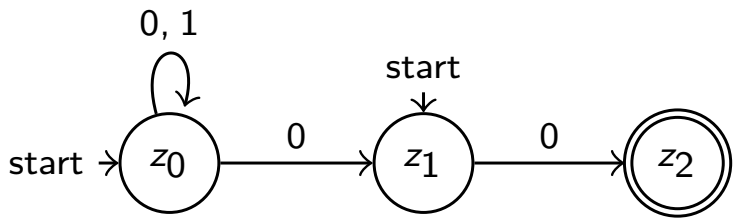

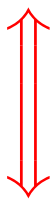

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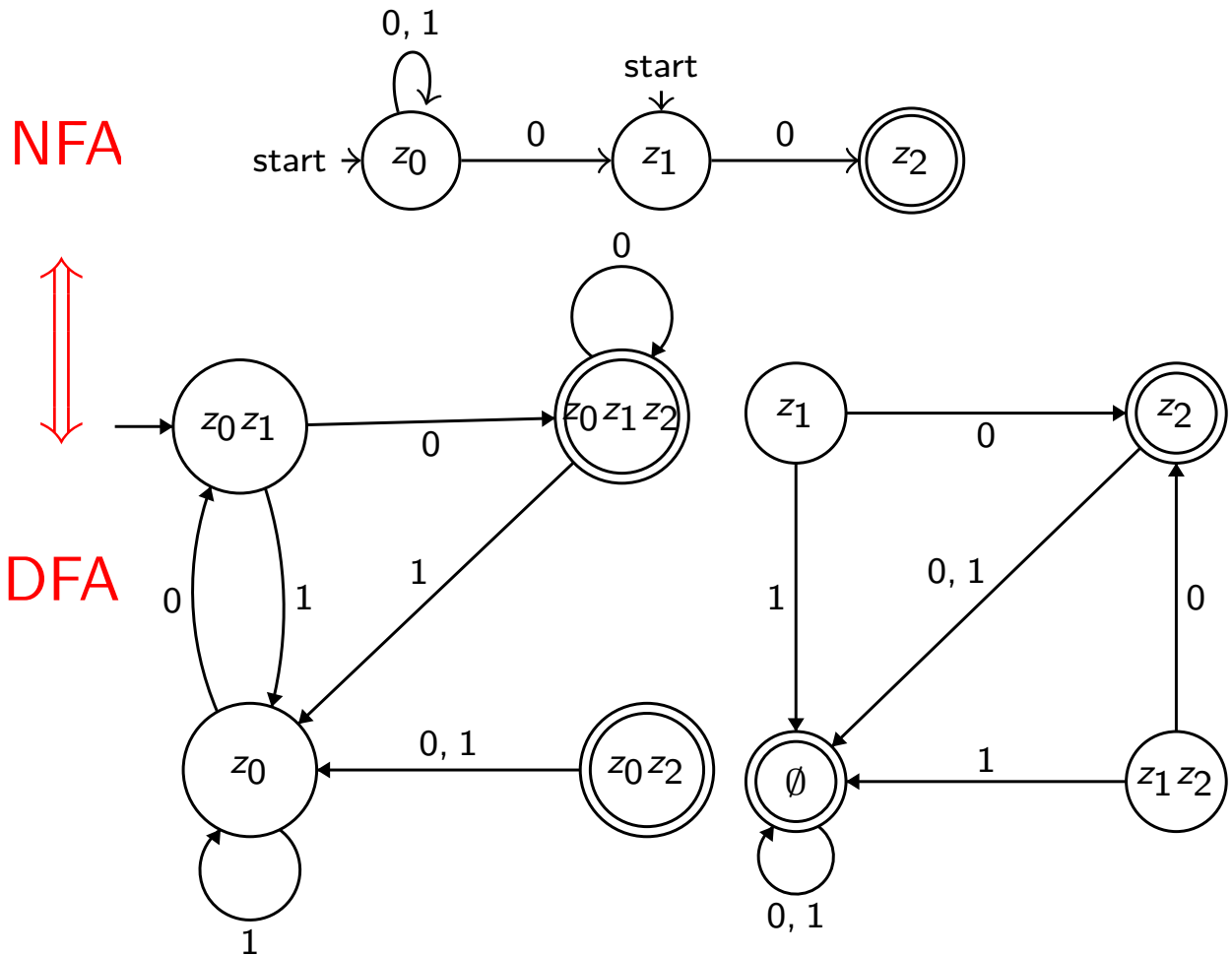

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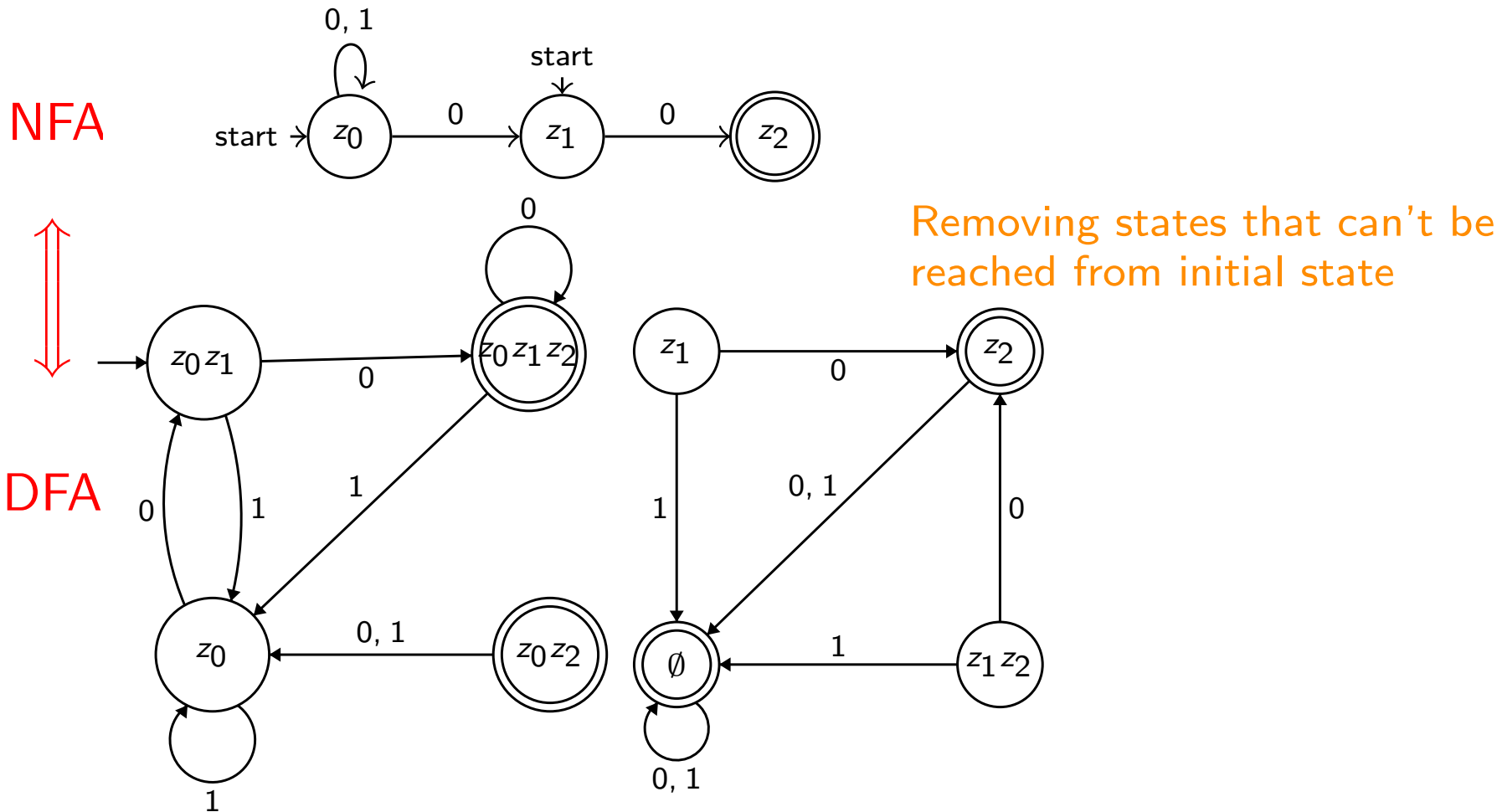

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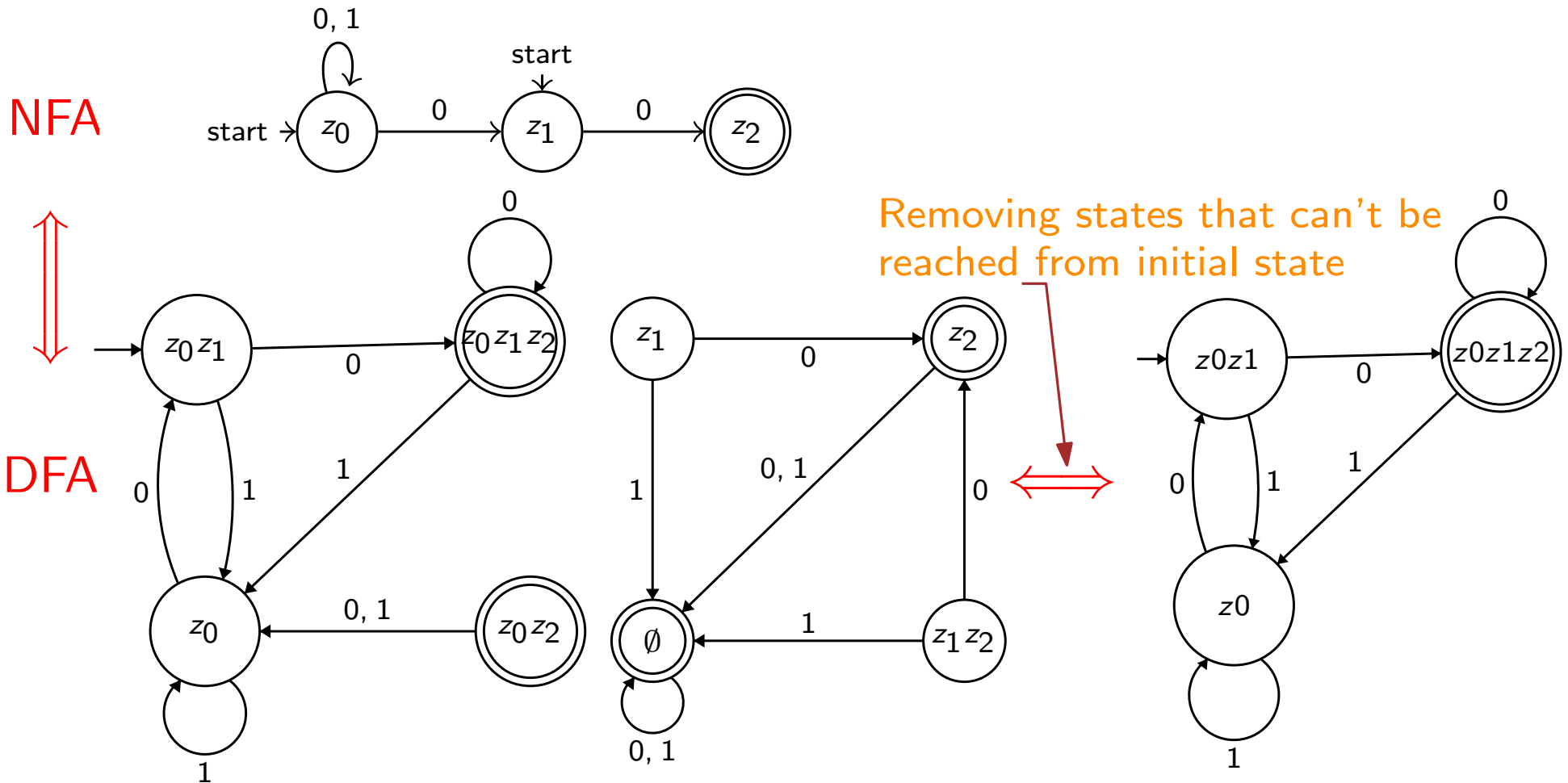

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$$\delta'(Z', a) = \bigcup_{z \in Z'} \delta(z, a) = \hat{\delta}(Z', a), \;\; Z' \in \mathcal{P}(Z)$$

$$Z_0' = S$$

$$E' = \{Z' \in \mathcal{P}(Z) \mid Z' \cap E \neq \emptyset\}$$

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$\Leftrightarrow$ $x \in L(M')$

Equivalence of DFA, NFA, Regular Grammars and Regular Expressions

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Theorem: DFA, NFA, regular grammars and regular expressions are equally expressive, meaning that they produce/recognize the same collection of languages.

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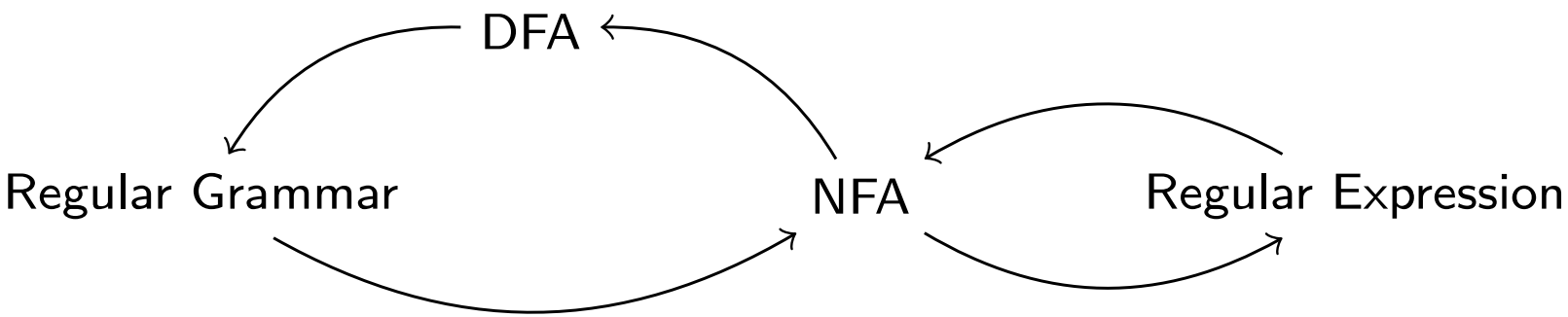

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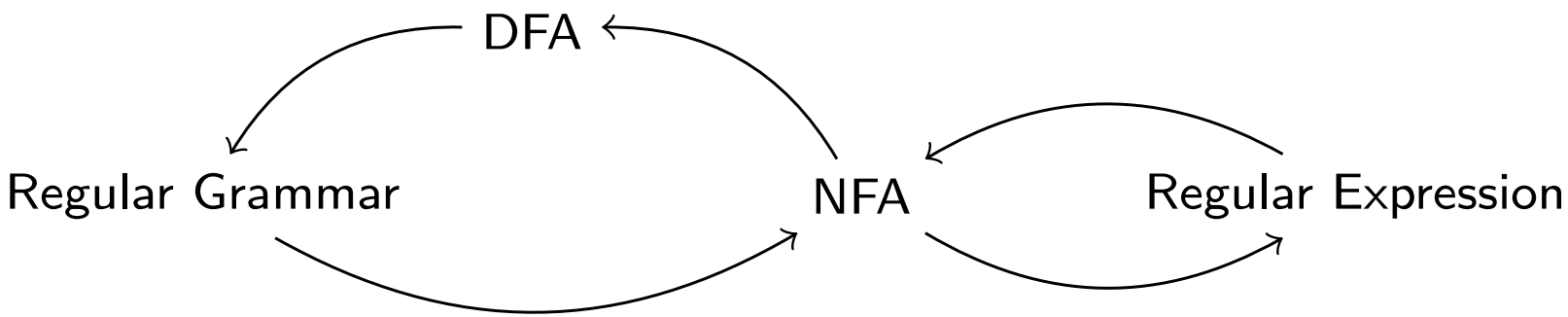

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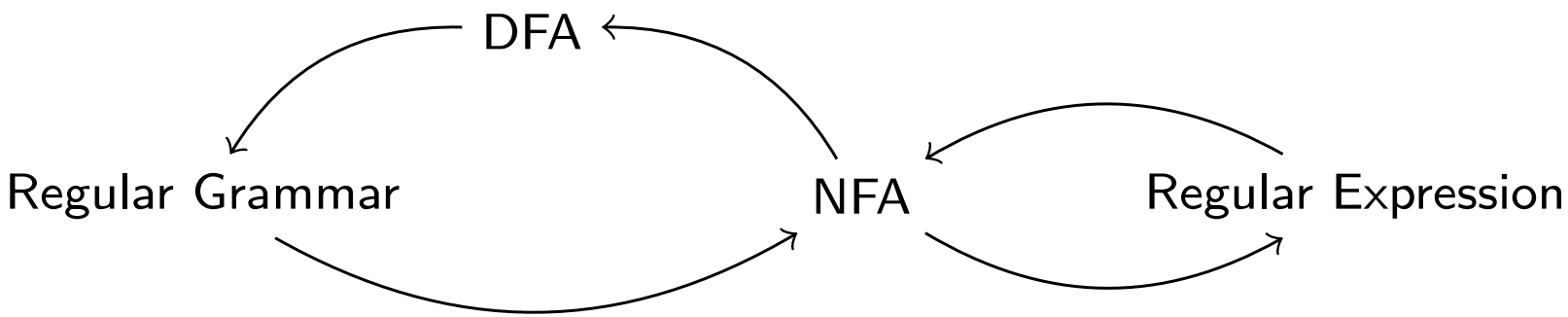

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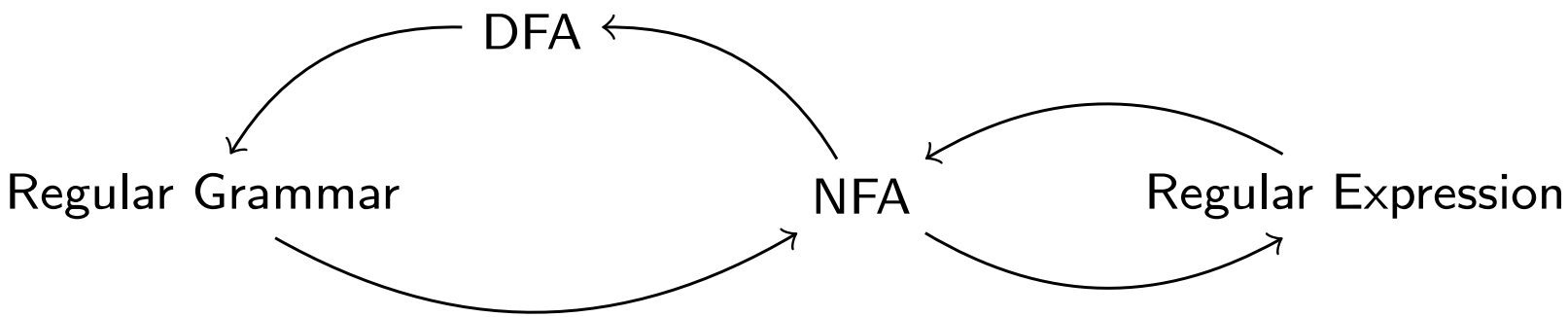

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- NFA $\leftrightarrow$ Regular expression

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Proof

$Z = V \cup \{X\}, X \notin V$

$S' = \{S\}$

$$E = \begin{cases} \{S, X\} & \text{if } (S \rightarrow \varepsilon) \in P \\ \{X\} & \text{else} \end{cases}$$

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Example

$S \rightarrow \mathrm{a}A \mid \mathrm{b}B$

$A \rightarrow \mathrm{a}B \mid \mathrm{a}C$

$B \rightarrow \mathrm{b}A \mid \mathrm{b}C$

$C \rightarrow \mathrm{c}C$

$A \rightarrow \mathrm{a}$

$B \rightarrow \mathrm{b}$

$C \rightarrow \mathrm{c}$

$S \rightarrow \varepsilon$

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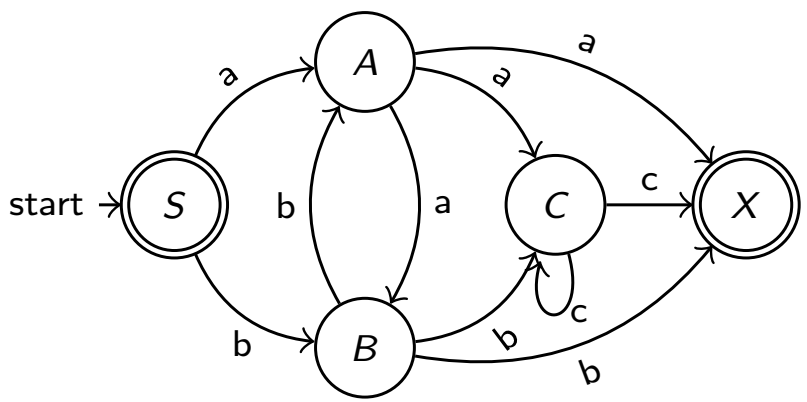
$B \rightarrow \mathrm{b}A \mid \mathrm{b}C$ $\quad$ $C \rightarrow \mathrm{c}$

$C \rightarrow \mathrm{c}C$ $\quad$ $S \rightarrow \varepsilon$



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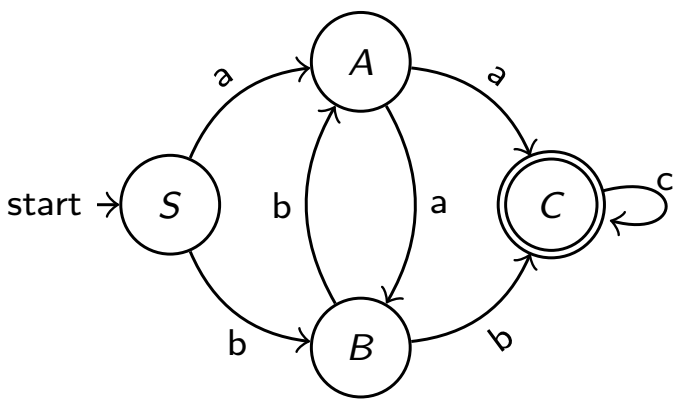

Example

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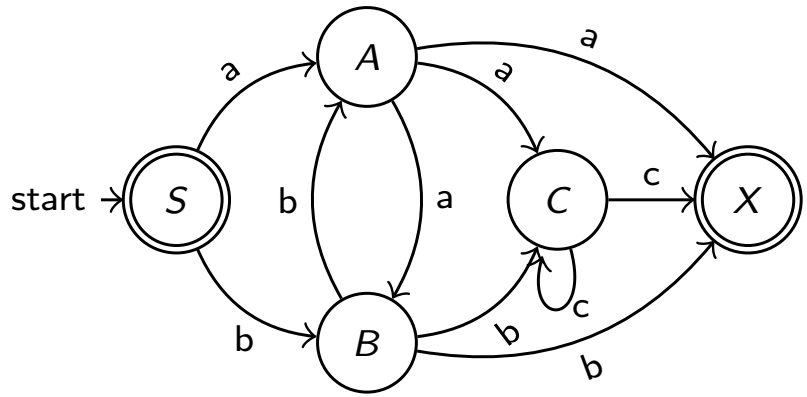
$A \rightarrow aB \mid aC$ $\quad B \rightarrow b$

$B \rightarrow bA \mid bC$ $\quad C \rightarrow c$

$C \rightarrow cC$ $\quad S \rightarrow \varepsilon$



Regular Grammar $\rightarrow$ NFA

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Theorem: For any regular expression R, the language $L(R)$ can be recognized by a NFA.

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Recall: Regular expressions

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$\emptyset$, ε and each $a \in \Sigma$ are regular expressions with $L(\emptyset) = \emptyset$, $L(\varepsilon) = \{\varepsilon\}$ and $L(a) = \{a\}$

If R_1 and R_2 are regular expressions then the following are also regular expressions:

- $(R_1 + R_2)$ with $L(R_1 + R_2) = L(R_1) \cup L(R_2)$
- $(R_1 R_2)$ with $L(R_1 R_2) = L(R_1)L(R_2)$
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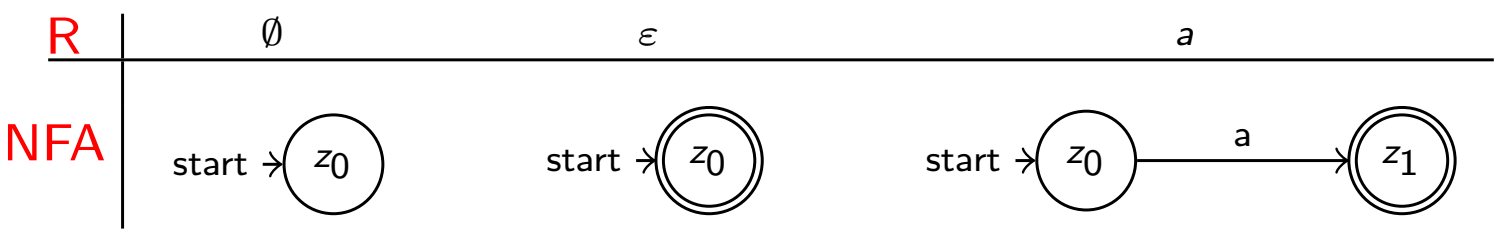

Regular Expressions $\rightarrow$ NFA

Proof (cont'd)

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$M = (Z_1 \cup Z_2, \Sigma, \delta_1 \cup \delta_2, S_1 \cup S_2, E_1 \cup E_2)$
and then have $L(M) = L(R)$.

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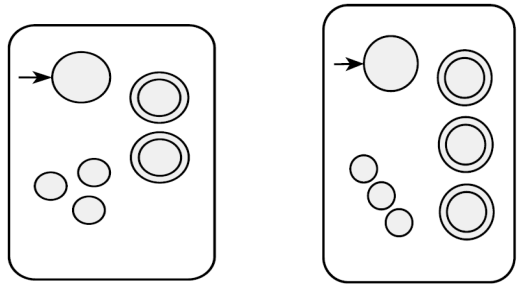

M_1 M_2

Regular Expressions $\rightarrow$ NFA

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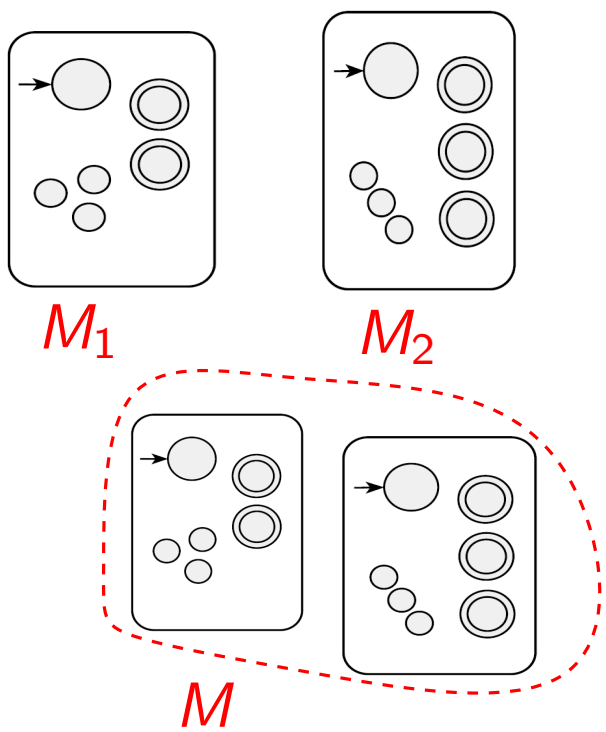

Regular Expressions $\rightarrow$ NFA

Proof (cont'd)

$R = R_1 R_2$

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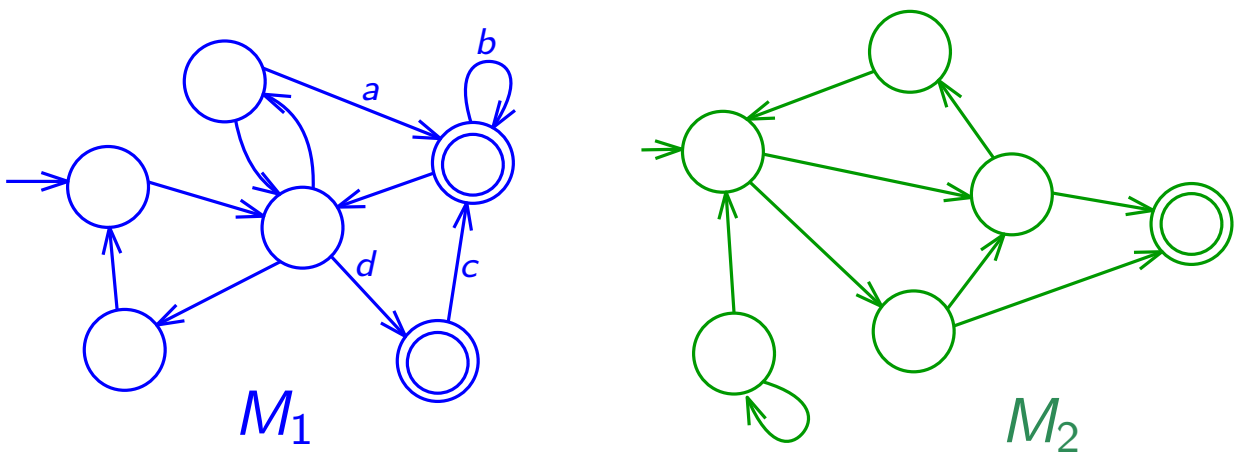

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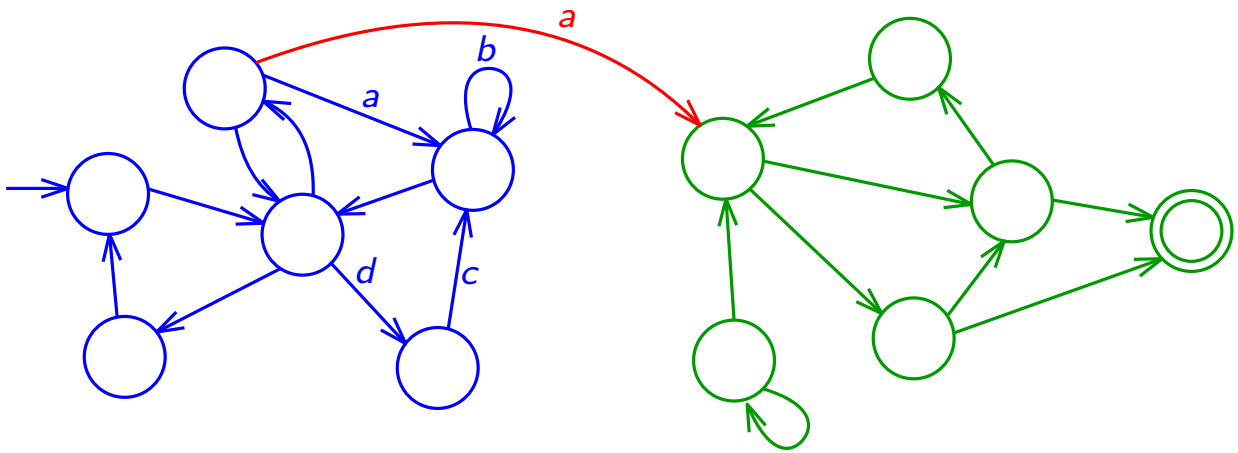

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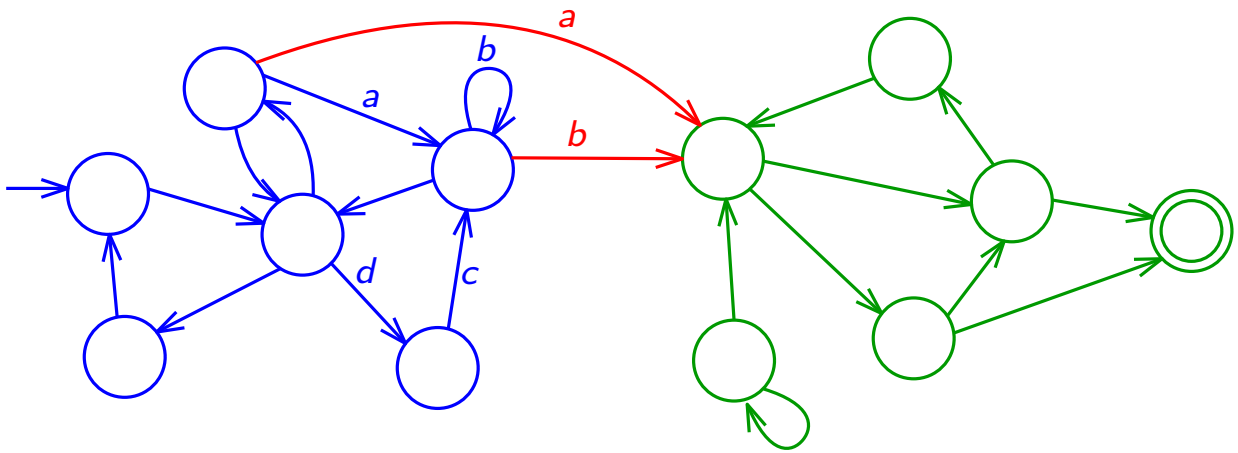

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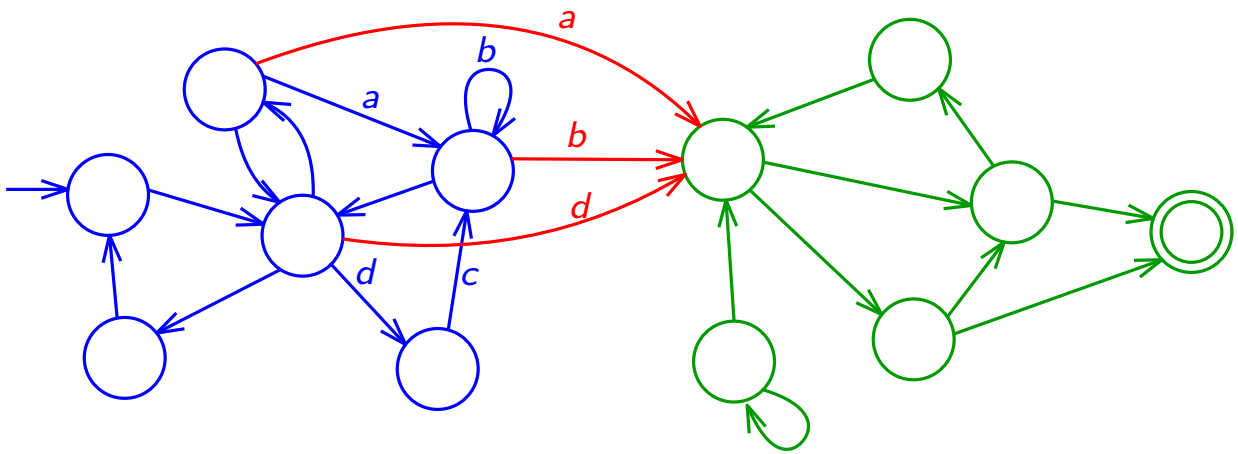

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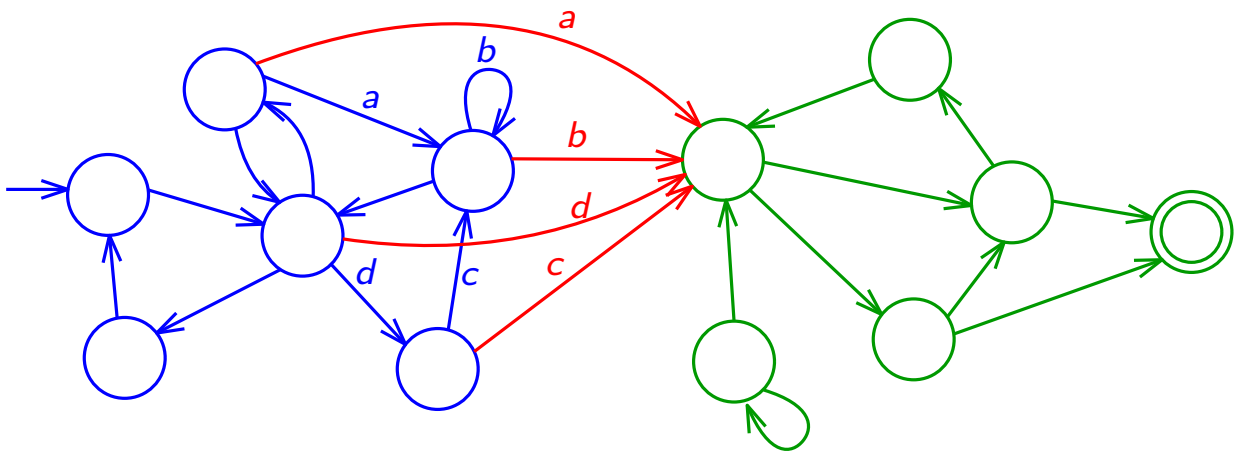

M

Regular Expressions $\rightarrow$ NFA

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Regular Expressions $\rightarrow$ NFA

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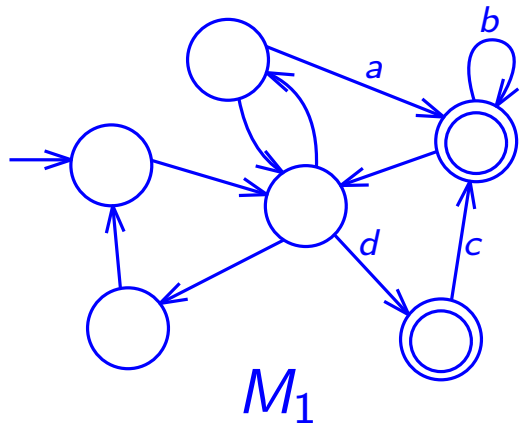

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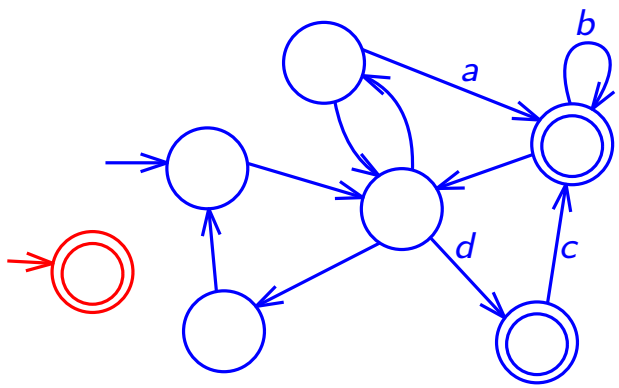

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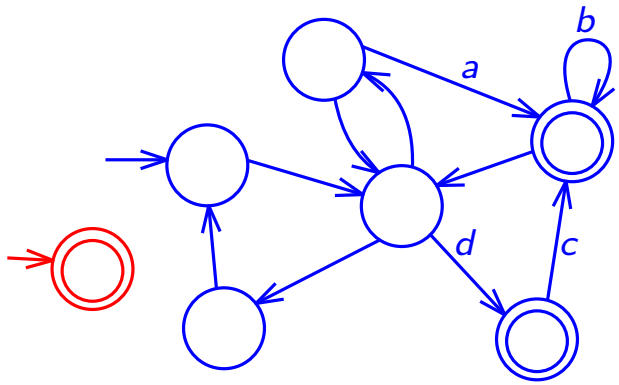

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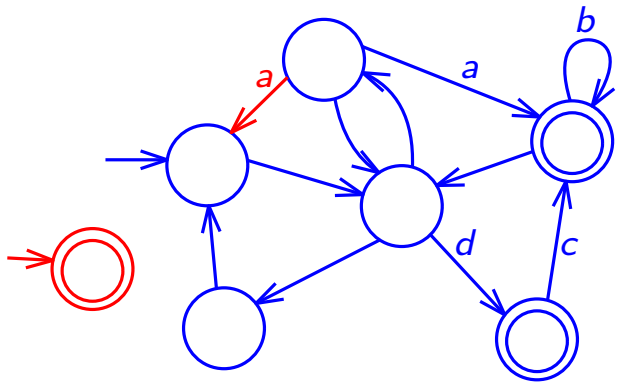

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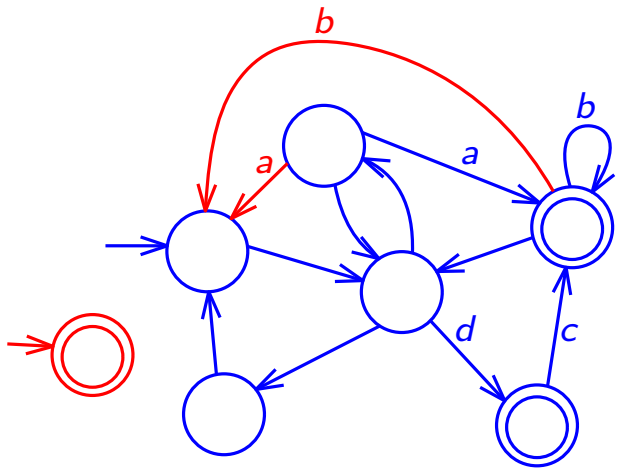

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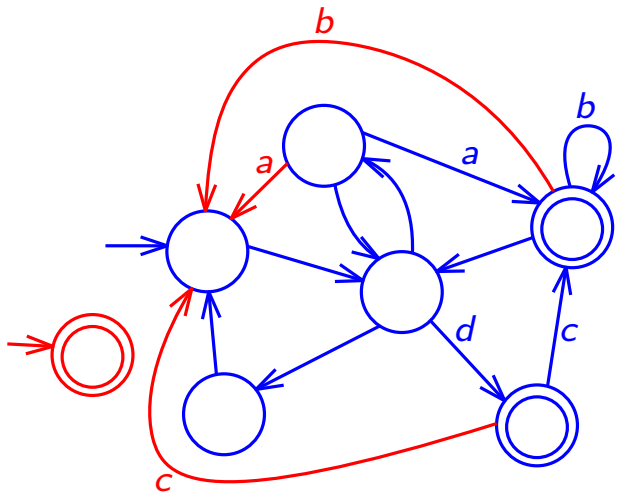

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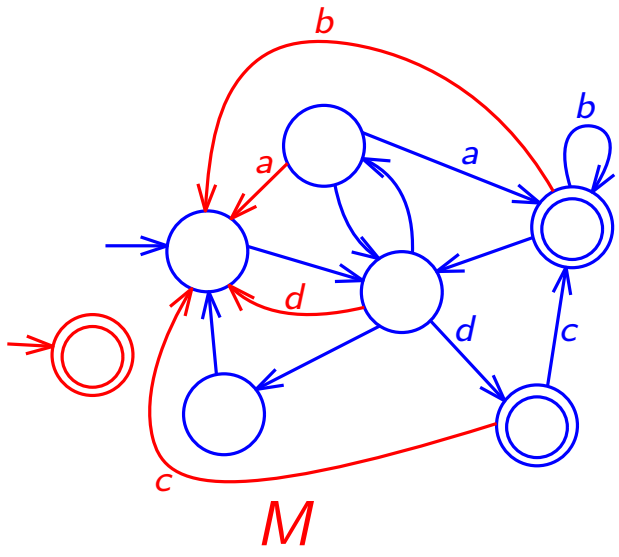

Constructing a DFA for a regular expression

Construct a DFA recognizing $(a+b)^*aba$.

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a

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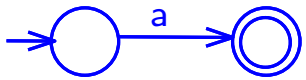
a 

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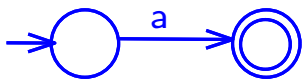
a 

b

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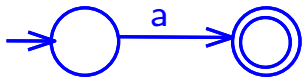
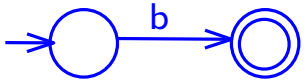
a 

b 

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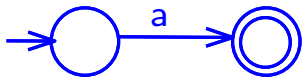
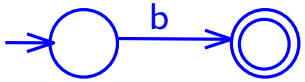
a 

b 

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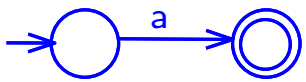
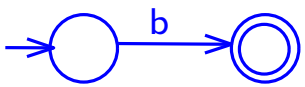
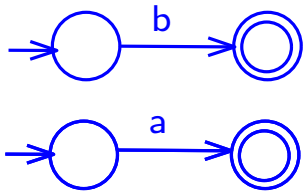
a 

b 

a+b 

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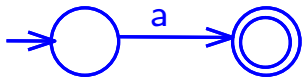
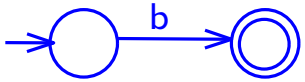
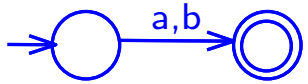
a 

b 

a+b 

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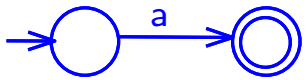
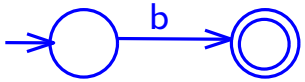
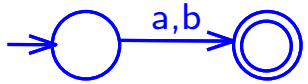
a 

b 

a+b 

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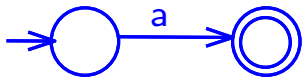
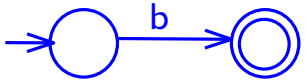
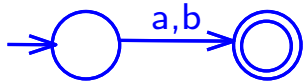
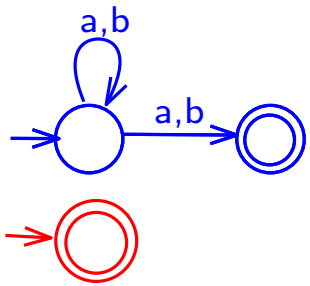
a 

b 

a+b 

$(a+b)^*$ 

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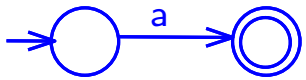
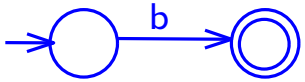
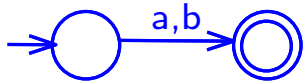
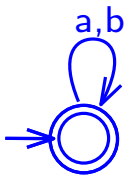
a 

b 

a+b 

$(a+b)^*$ 

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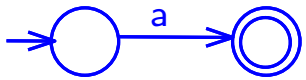
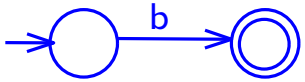
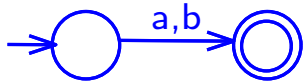
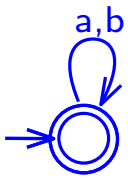
a 

b 

a+b 

$(a+b)^*$ 

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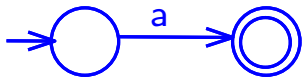
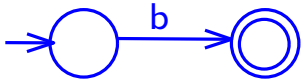
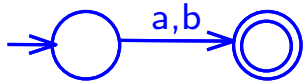
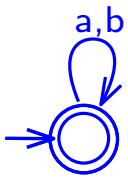
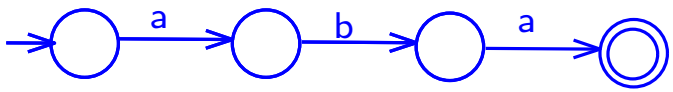
a 

b 

a+b 

$(a+b)^*$ 

aba 

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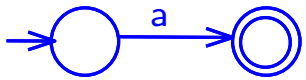
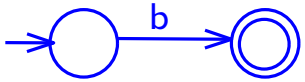
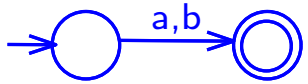
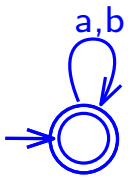
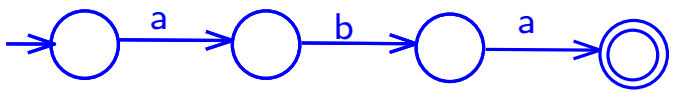
a 

b 

a+b 

$(a+b)^*$ 

aba 

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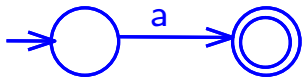
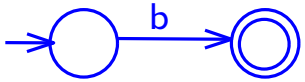
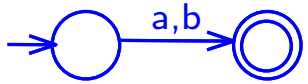
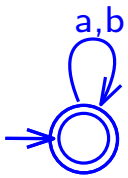
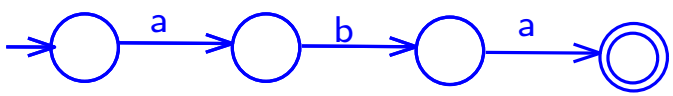
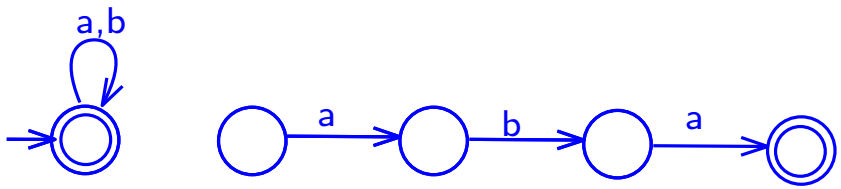
a 

b 

a+b 

$(a+b)^*$ 

aba 

$(a+b)^*aba$ 

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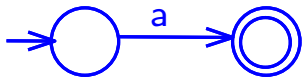
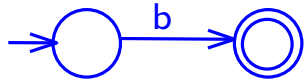
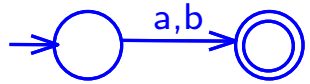
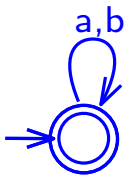
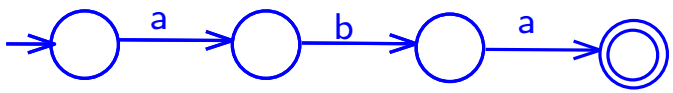
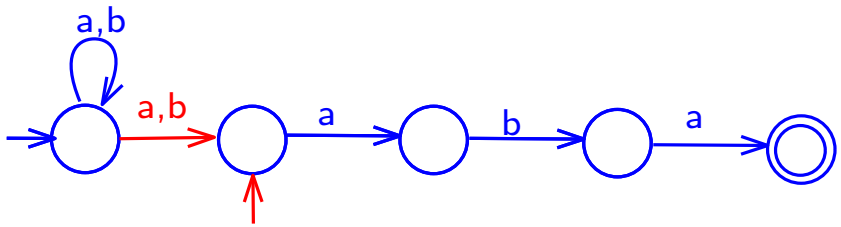
a 

b 

a+b 

$(a+b)^*$ 

aba 

$(a+b)^*aba$ 

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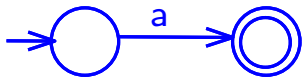
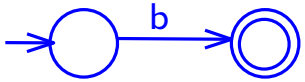
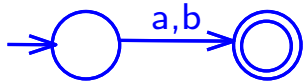
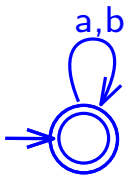
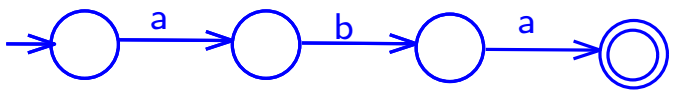
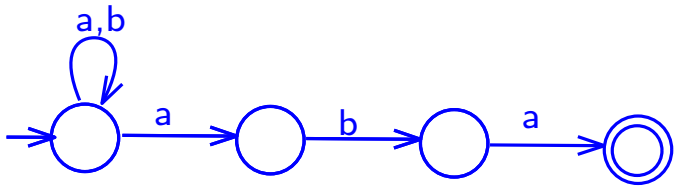
a 

b 

a+b 

$(a+b)^*$ 

aba 

$(a+b)^*aba$ 

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Theorem: Any language which can be recognized by an NFA can be described by a regular expression.

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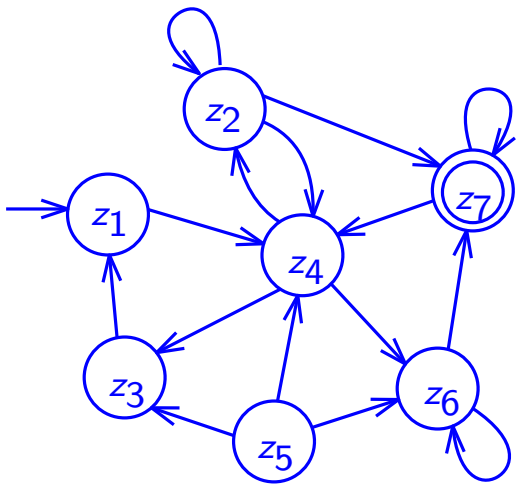

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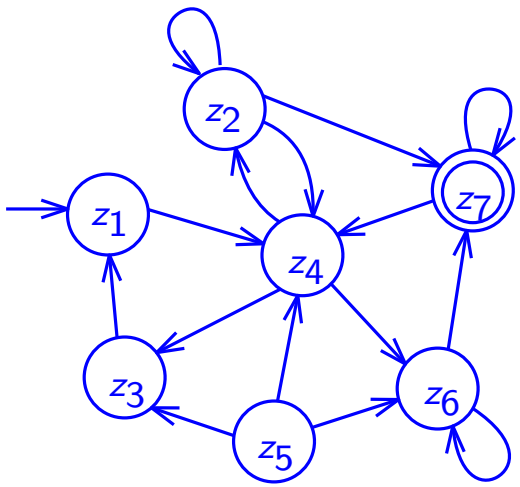

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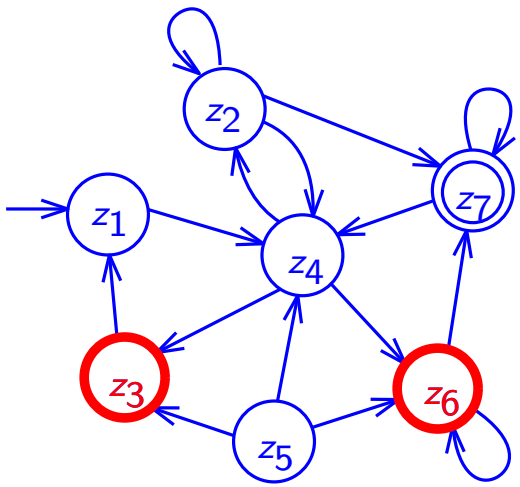


NFA $\rightarrow$ Regular Expressions

Theorem: Any language which can be recognized by an NFA can be described by a regular expression.

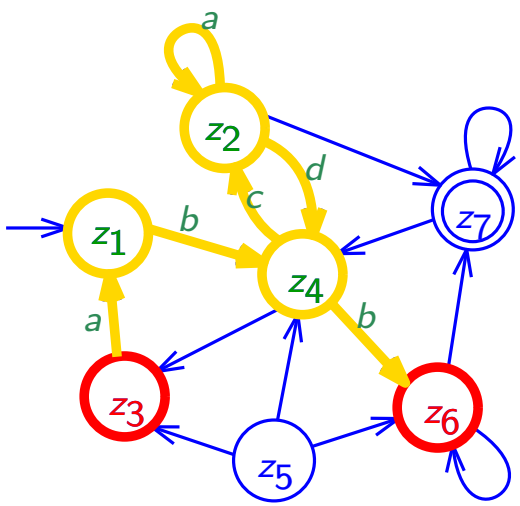
Proof

Let M be a DFA with $L = L(M)$ and let the states be $Z = \{z_1, \ldots, z_n\}$ where z_1 is the initial state.

For $i, j \in \{1, \ldots, n\}$ and $k \in \{0, 1, \ldots, n\}$ we define languages $L^k_{i,j}$ and show by induction on k that they can be generated by regular expressions.

$$L^k_{i,j} := \{x \in \Sigma^* \mid \text{on input } x \text{ } M \text{ starts in } z_i \text{ and ends in } z_j \ (\hat{\delta}(z_i, x) = z_j)$$
$$\text{while the intermediate states (despite } z_i, z_j) \text{ have index} \leq k\}$$

$L^5_{3,6}$?



NFA $\longrightarrow$ Regular Expressions

Proof (cont'd)

Base of induction: For $k = 0$ we have

$$i \neq j: \quad L^0_{i,j} = \{a \in \Sigma \mid \delta(z_i, a) = z_j\}$$

NFA $\longrightarrow$ Regular Expressions

Proof (cont'd)

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$$L^0_{i,i} = \{\varepsilon\} \cup \{a \in \Sigma \mid \delta(z_i, a) = z_i\}$$

NFA $\longrightarrow$ Regular Expressions

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Thus $L^0_{i,j}$ is finite and can be generated by a regular expression (i.e. just a sequence of $+$ expressions).

NFA $\rightarrow$ Regular Expressions

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Thus $L^0_{i,j}$ is finite and can be generated by a regular expression (i.e. just a sequence of $+$ expressions).

Induction step: Let $k \geq 1$ and $L^k_{i,j}$ be expressible by a regular expression ($L^k_{i,j} = L(R^k_{i,j})$).

NFA $\rightarrow$ Regular Expressions

Proof (cont'd)

To use the induction step, we need to express $L_{i,j}^{k+1}$ in terms of those with smaller upper indices.

NFA $\rightarrow$ Regular Expressions

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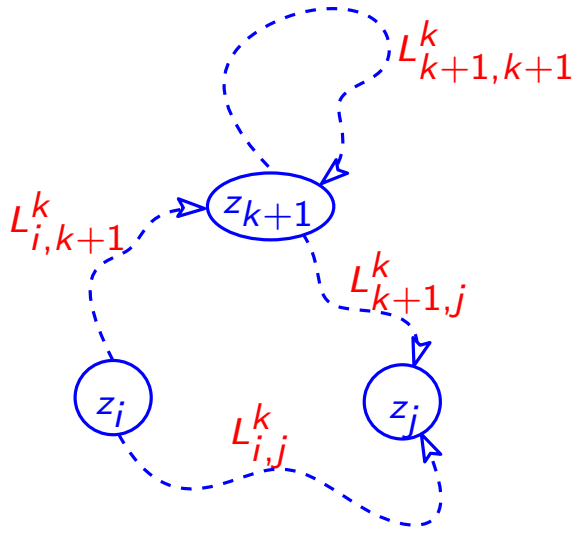
$$L_{i,j}^{k+1} = L_{i,j}^{k} \cup L_{i,k+1}^{k} (L_{k+1,k+1}^{k})^* L_{k+1,j}^{k}$$



NFA $\rightarrow$ Regular Expressions

Proof (cont'd)

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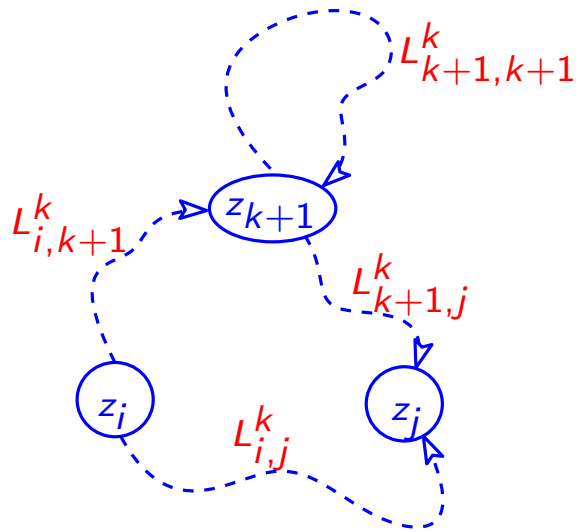

$$L_{i,j}^{k+1} = L_{i,j}^{k} \cup L_{i,k+1}^{k} (L_{k+1,k+1}^{k})^* L_{k+1,j}^{k}$$

Explanation: If z_{k+1} is not needed to travel from z_i to z_j we have $L_{i,j}^k$. Otherwise we run trough z_{k+1} one or multiple times, but we start and stop at z_{k+1} for the substring in the middle.

NFA $\rightarrow$ Regular Expressions

Proof (cont'd)

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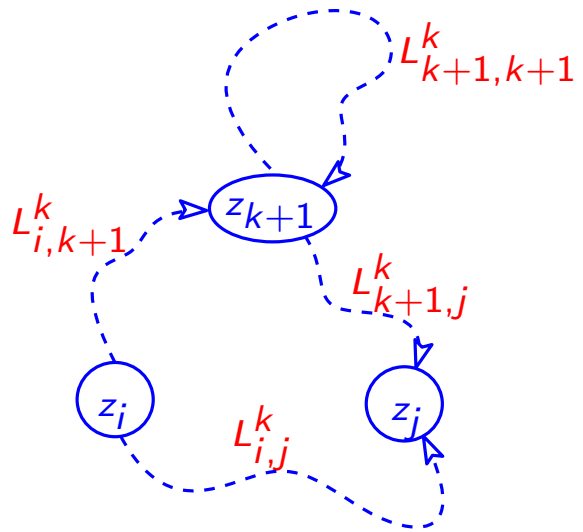

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By the induction hypothesis, $L_{i,j}^k$, $L_{i,k+1}^k$, $L_{k+1,k+1}^k$ and $L_{k+1,j}^k$ can be generated by the regular expression and thus $L_{i,j}^{k+1}$ can be generated by

$$R_{i,j}^{k+1} = R_{i,j}^{k} + R_{i,k+1}^{k}(R_{k+1,k+1}^{k})^* R_{k+1,j}^{k}.$$

NFA $\rightarrow$ Regular Expressions

Proof (cont'd)

Finally we have

$$L(M) = \bigcup_{z_i \in E} L^n_{1,i}$$

NFA $\rightarrow$ Regular Expressions

Proof (cont'd)

Finally we have

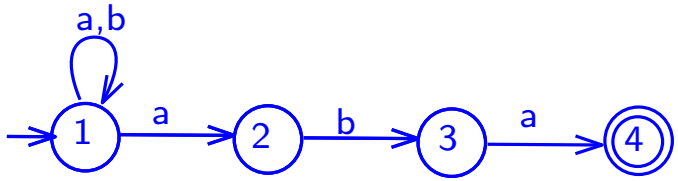
$$L(M) = \bigcup_{z_i \in E} L^n_{1,i}$$

If $i_1, i_2, \ldots, i_m$ are the indices of the final states we have

$$R = (R^n_{1,i_1} + R^n_{1,i_2} + \cdots + R^n_{1,i_m})$$

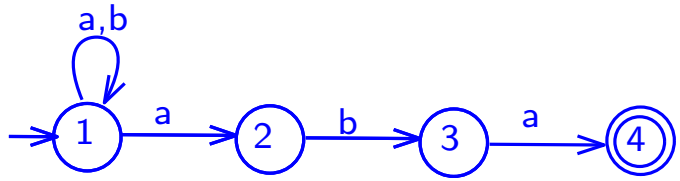
with $L(R) = L(M)$, which completes the proof.

NFA $\longrightarrow$ Regular Expressions



M

NFA $\rightarrow$ Regular Expressions



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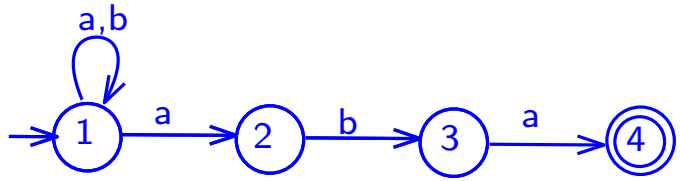
We need to find $R = (R^n_{1,i_1} + R^n_{1,i_2} + \cdots + R^n_{1,i_m})$ with $i_1, i_2, \ldots, i_m$ being the final states.

NFA $\rightarrow$ Regular Expressions



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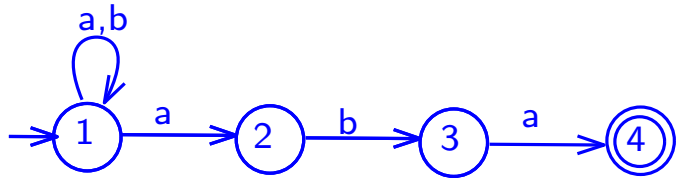
NFA $\rightarrow$ Regular Expressions



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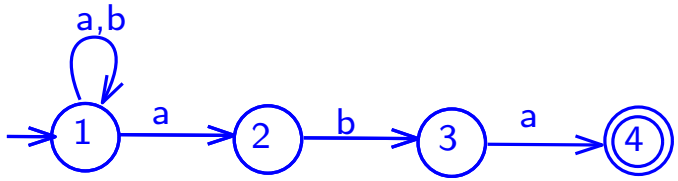
Remeber: $R^{k+1}_{i,j} = R^k_{i,j} + R^k_{i,k+1}(R^k_{k+1,k+1})^* R^k_{k+1,j}$.

NFA $\rightarrow$ Regular Expressions



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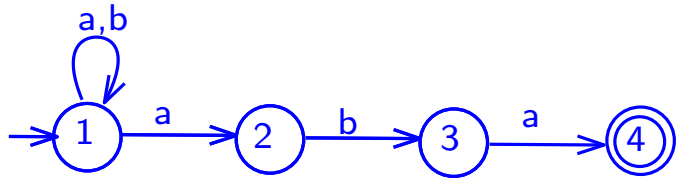
$R^0_{1,1} = (\varepsilon + a + b)$

NFA $\rightarrow$ Regular Expressions



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$R^1_{1,2} = (a+b)^* a$

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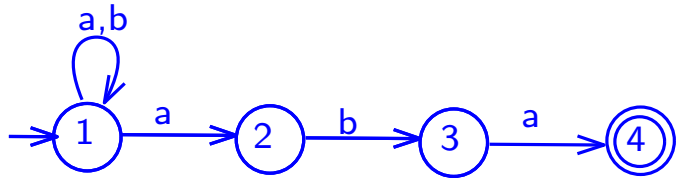
NFA $\rightarrow$ Regular Expressions



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$R^1_{1,2} = (a+b)^* a$

$R^2_{1,3} = (a+b)^* ab$

$R^3_{1,4} = (a+b)^* aba$

Summary

The following are equally expressive in the sense that they all produce/recognize regular languages:

- Regular Grammars
- NFAs (Nondeterministic Finite Automata)
- DFAs (Deterministic Finite Automata)
- Regular Expressions