Automata Theory and Formal Languages

Summer Term 2025

8th Lecture

ε-FA

Prof. Dr. Matthias Mnich *Institute for Algorithms and Complexity*

Regular Languages

Summary of Previous Lecture

The following are equivalent in the sense that they all produce/recognize regular languages:

- Regular Expressions
- Regular Grammars
- DFAs (Deterministic Finite Automata)
- NFAs (Nondeterministic Finite Automata)

ε-FA

ε-finite automata

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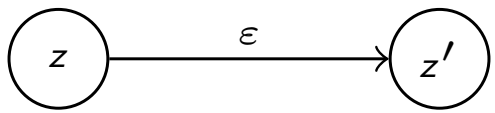

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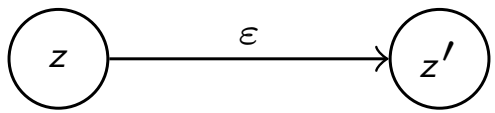

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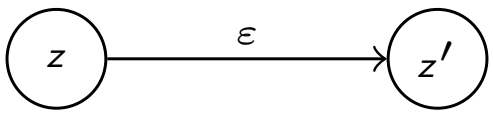

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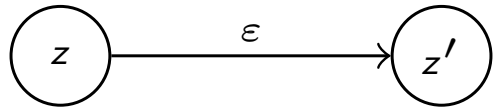

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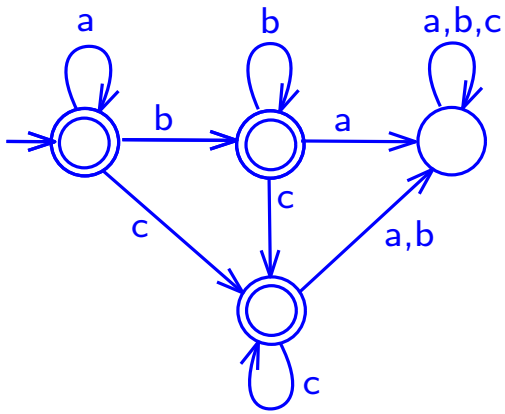

A DFA recognizing a*b*c*

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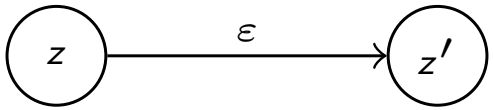

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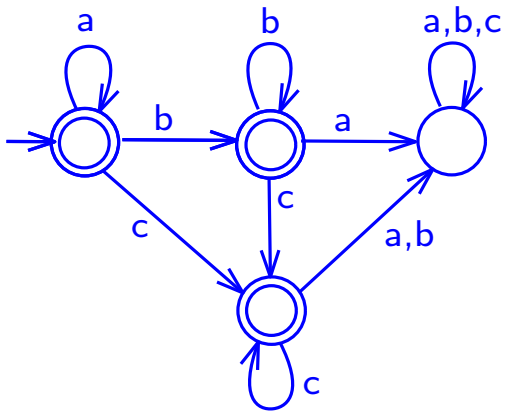

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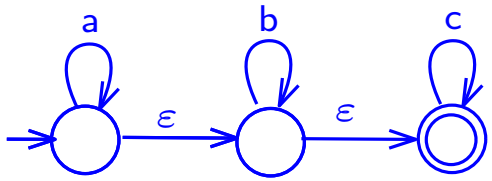
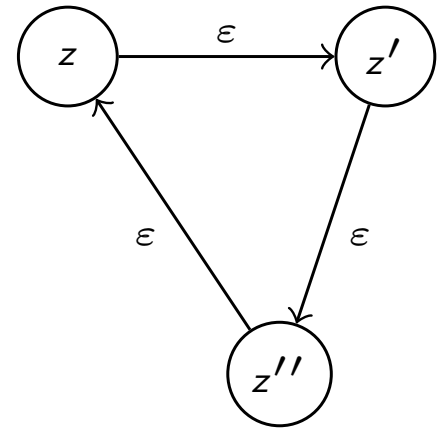


An ε-FA recognizing a*b*c*

Formal definition of ε-finite automata

ε-cycle



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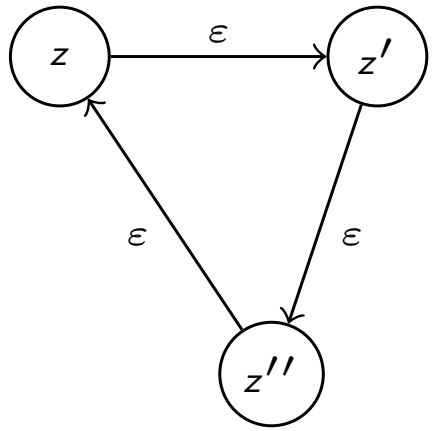
Such an ε-cycle can be replaced by a single state



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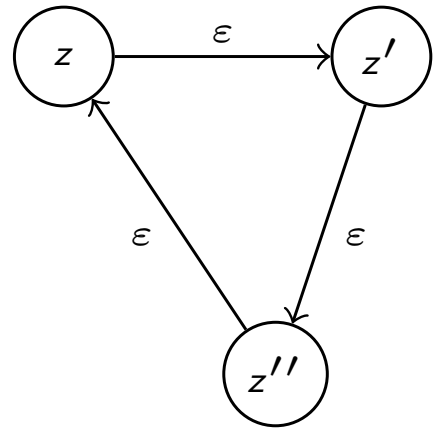

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Definition

An ε-FA (or an NFA with ε-transitions) M is a 5 tuple

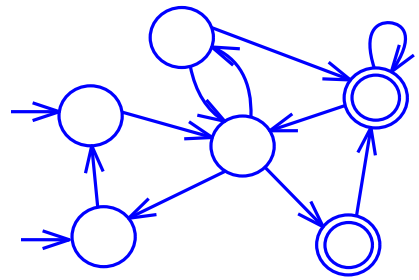
$$M = (Z, \Sigma, \delta, S, E)$$

where

- δ: $Z \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Z)$ Transition function that has no ε-cycles
- Z, Σ, S, E are as NFAs

Regular operations with ε-FA

- Multiple starting/final states
  $\rightarrow$ single starting/final state



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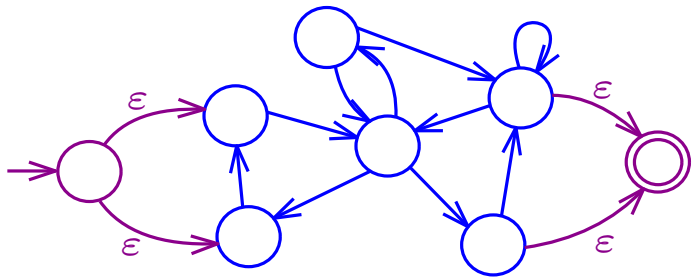

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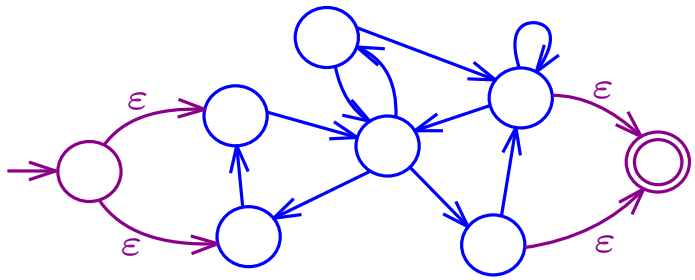

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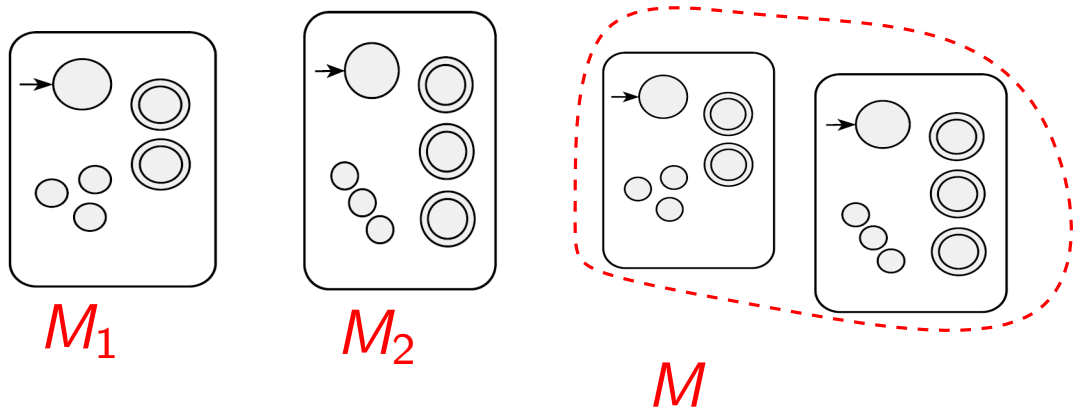
$$L(M) = L(M_1) \cup L(M_2)$$



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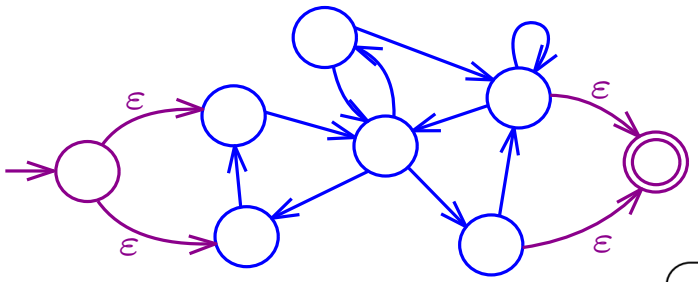

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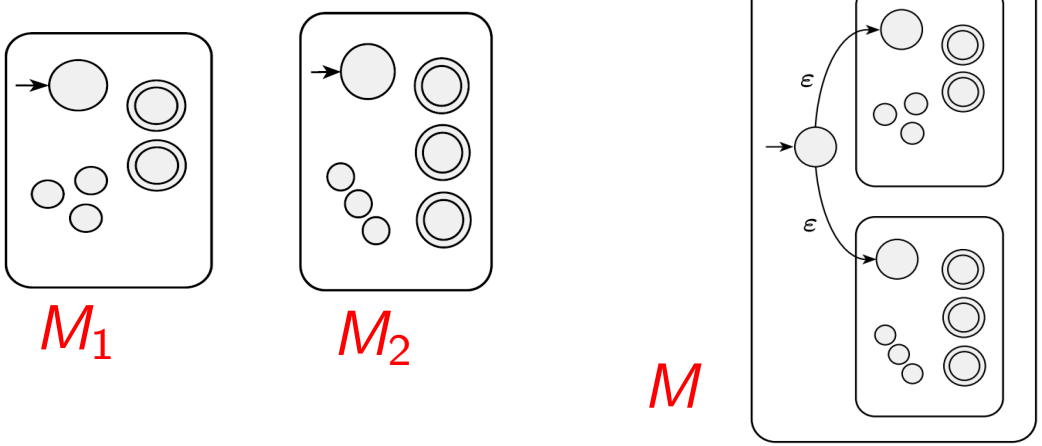

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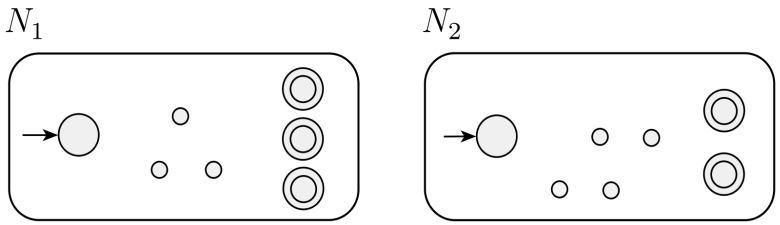

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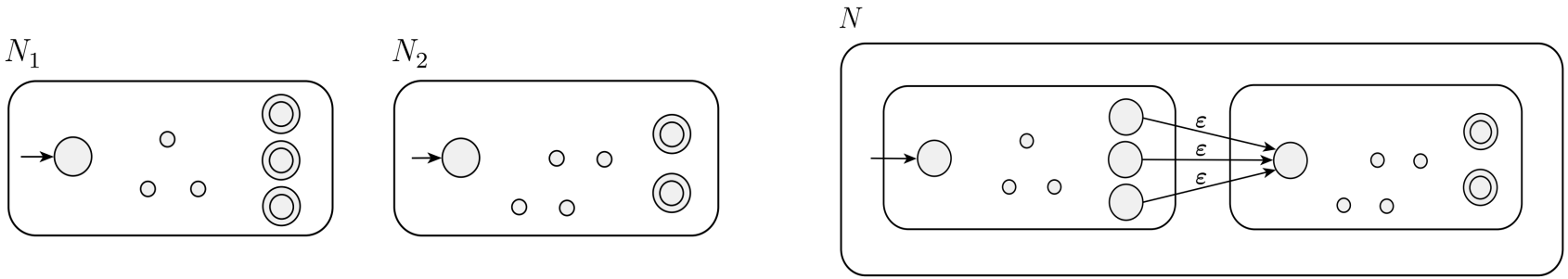

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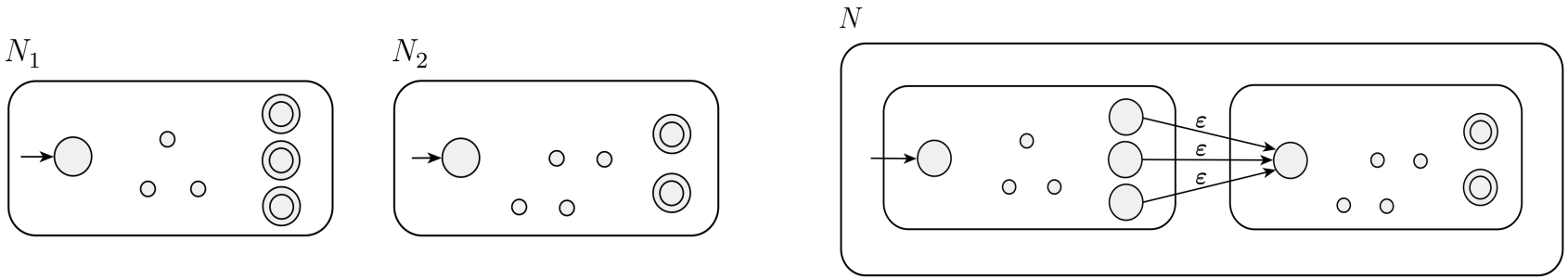

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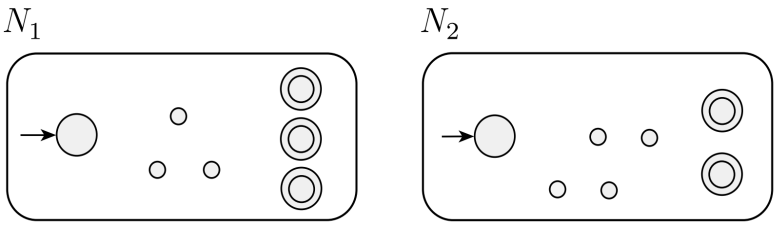

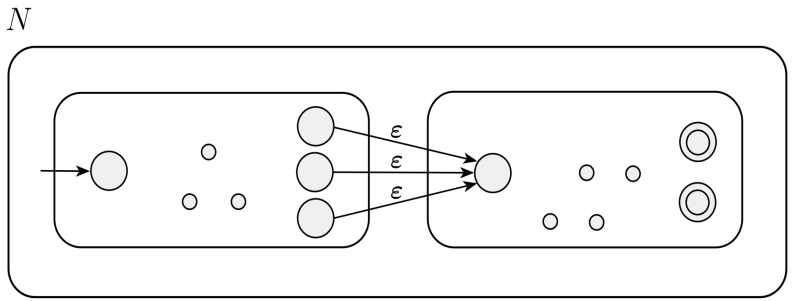

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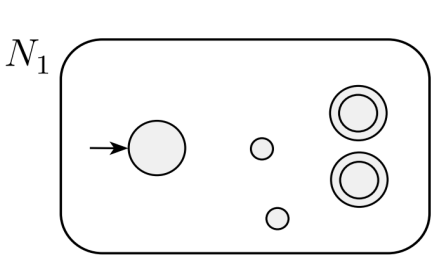

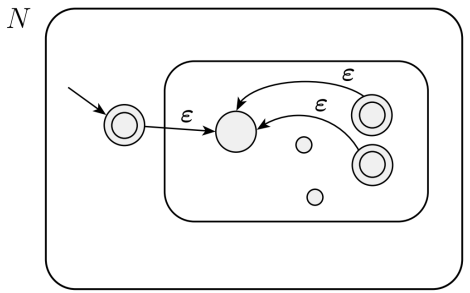

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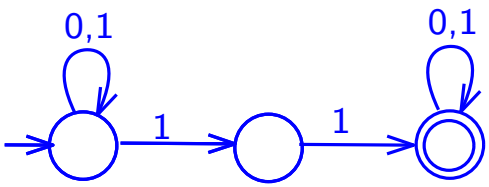

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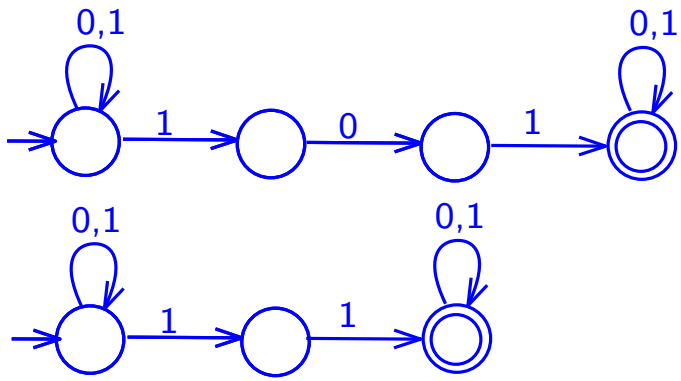

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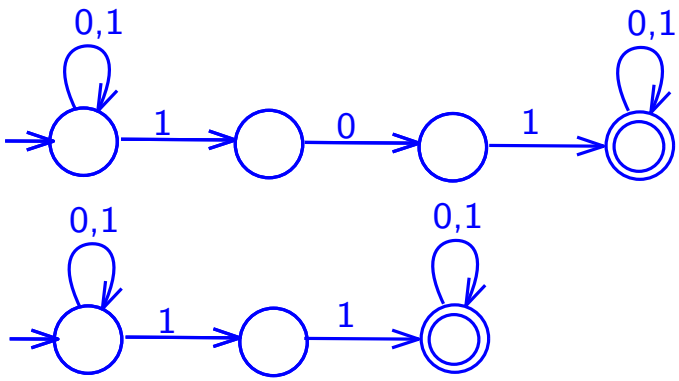

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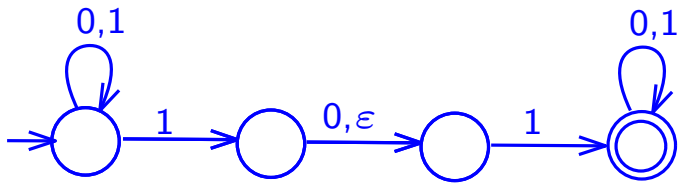

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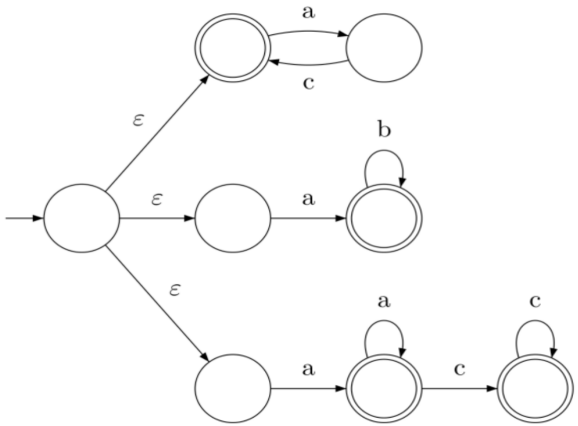

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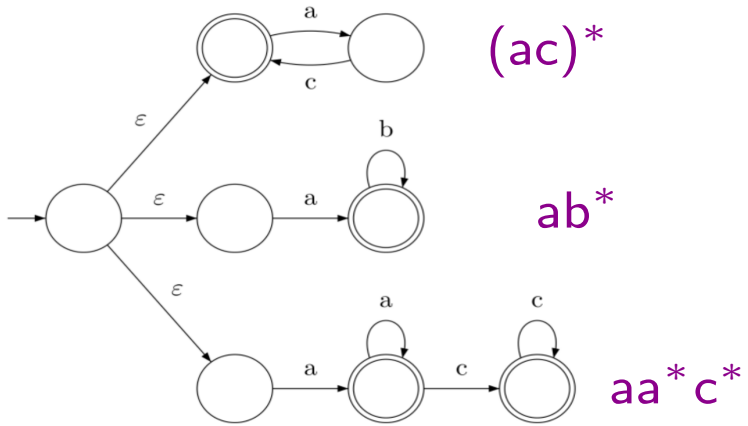

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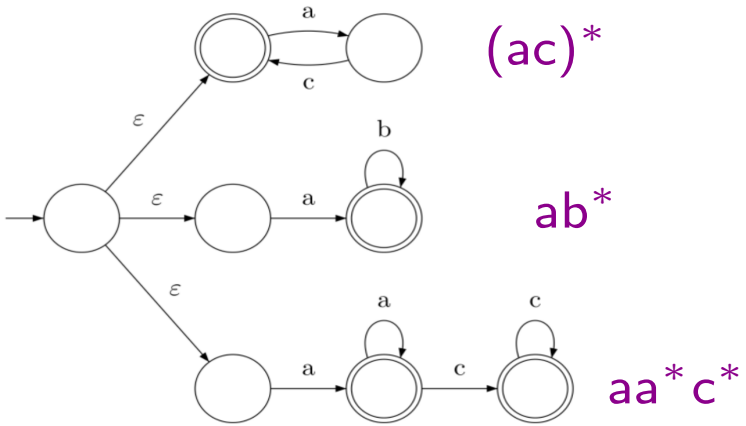

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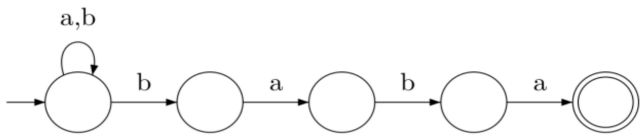

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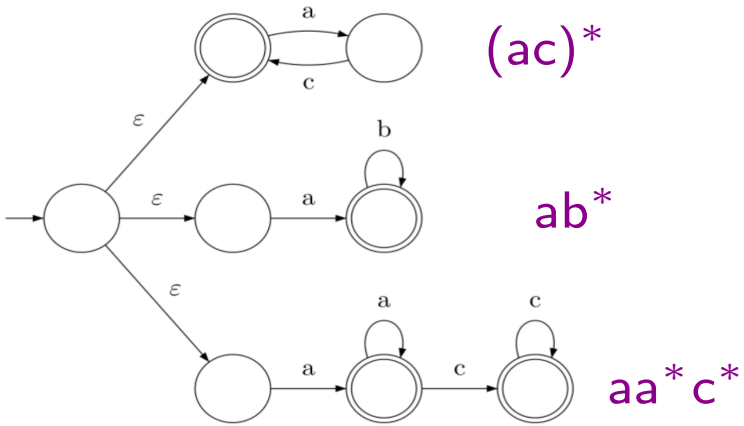

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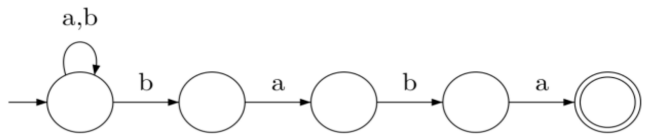

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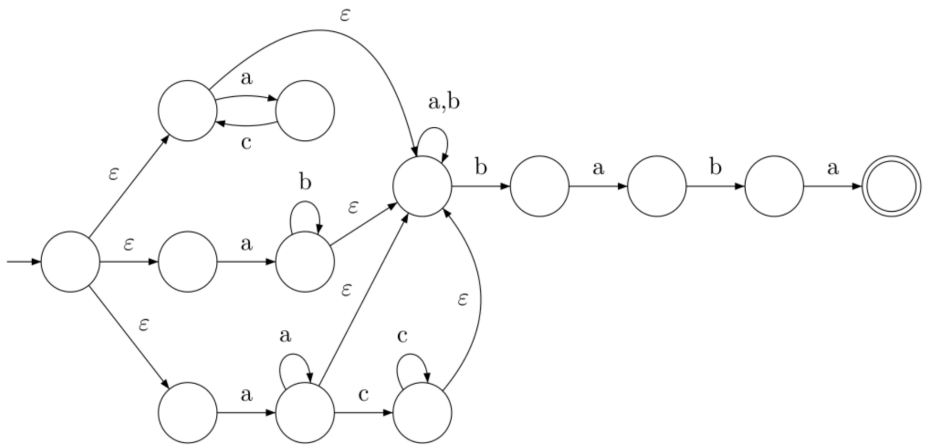

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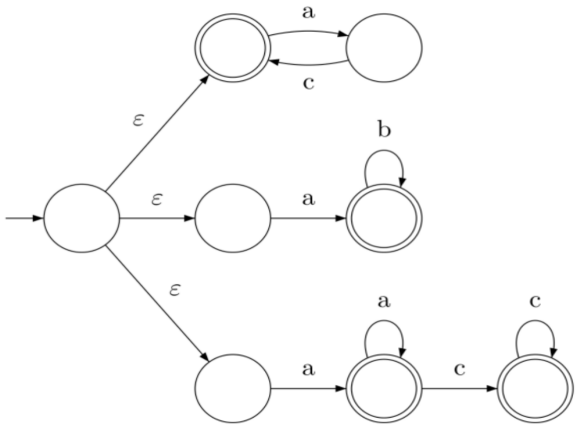

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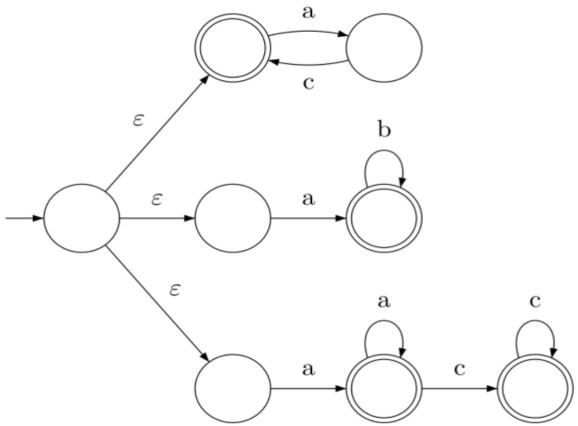

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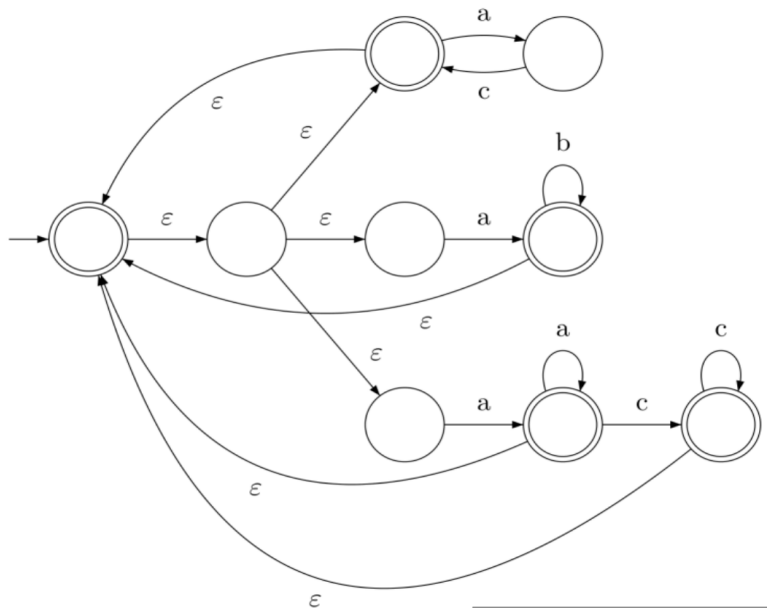

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Converting ε-FA to NFA

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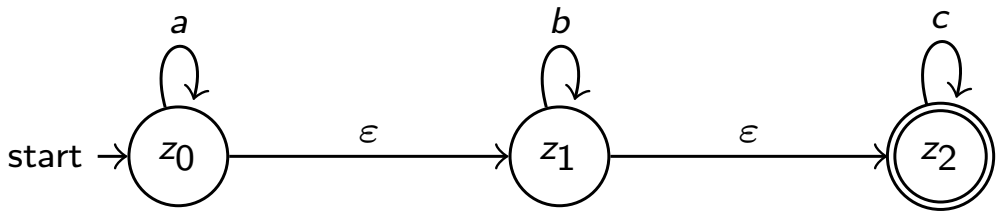

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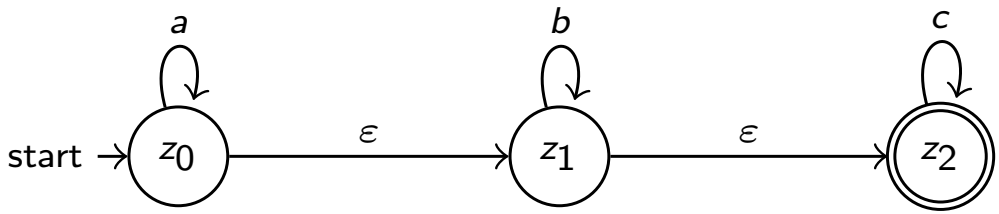

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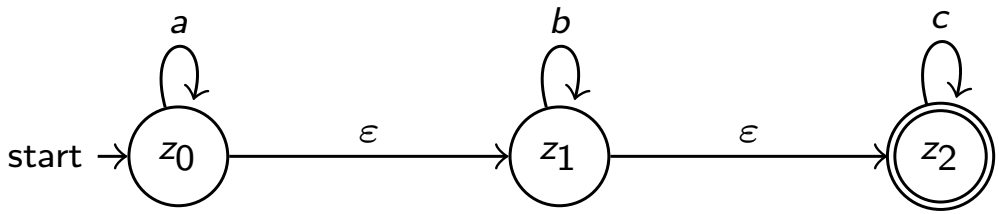

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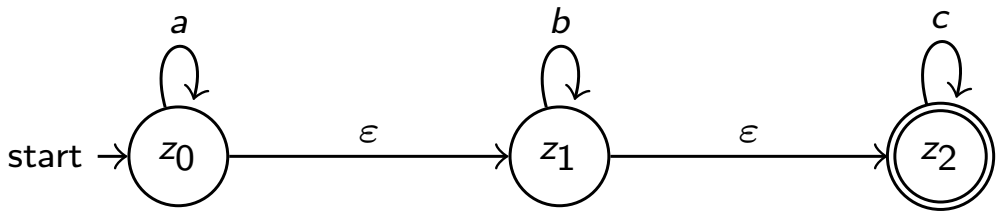

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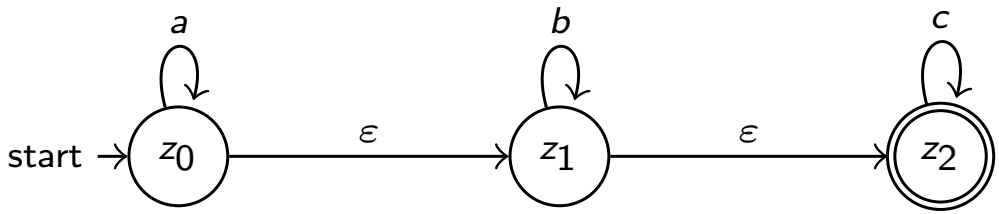

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- if $\varepsilon\text{-closure}(z) = \{q_1, \ldots, q_r\}$, then
$$\delta'(z, a) := \varepsilon\text{-closure}\big(\delta(q_1, a)\big) \cup \cdots \cup \varepsilon\text{-closure}\big(\delta(q_r, a)\big).$$

Converting ε-FA to NFA

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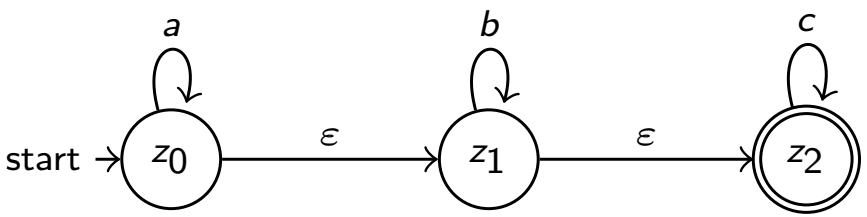

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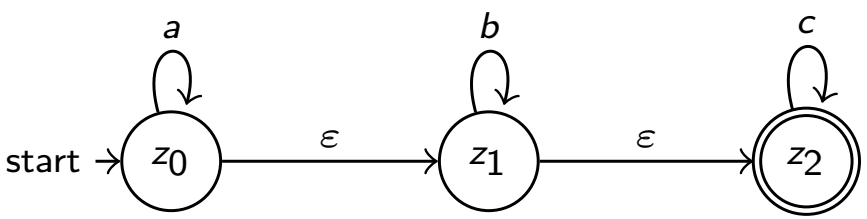

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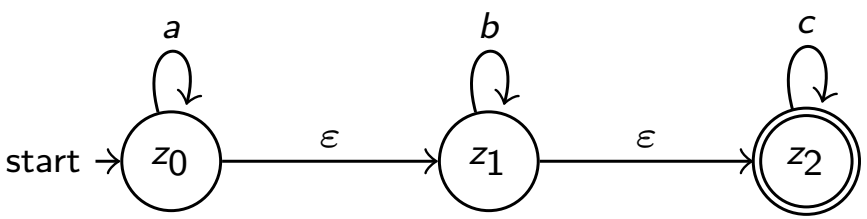

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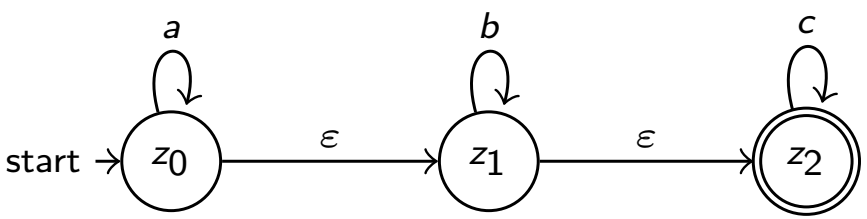

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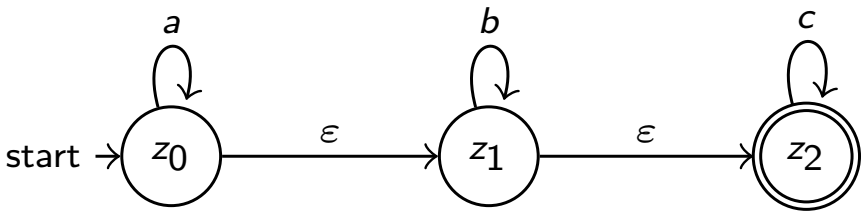

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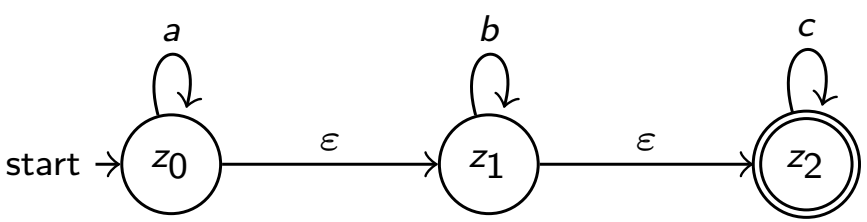

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Then $M' = (Z, \Sigma, \delta', S, E')$ is an equivalent NFA with M.

Theorem

Each ε-FA can be converted into an equivalent NFA.

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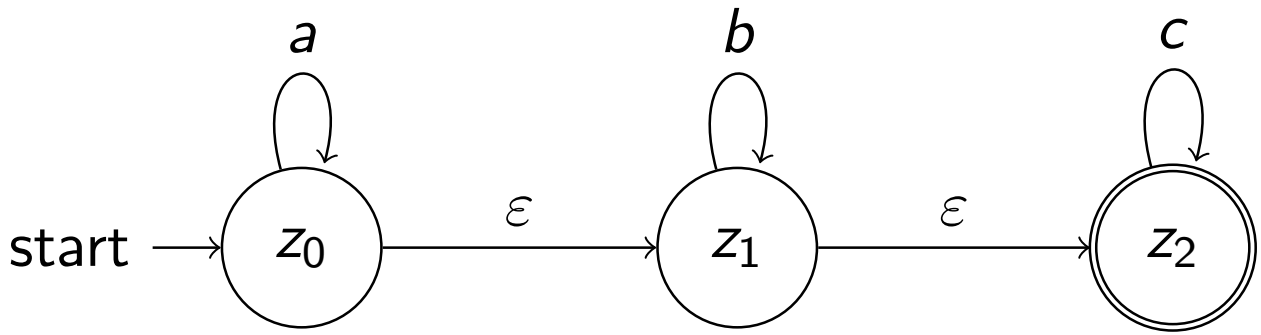

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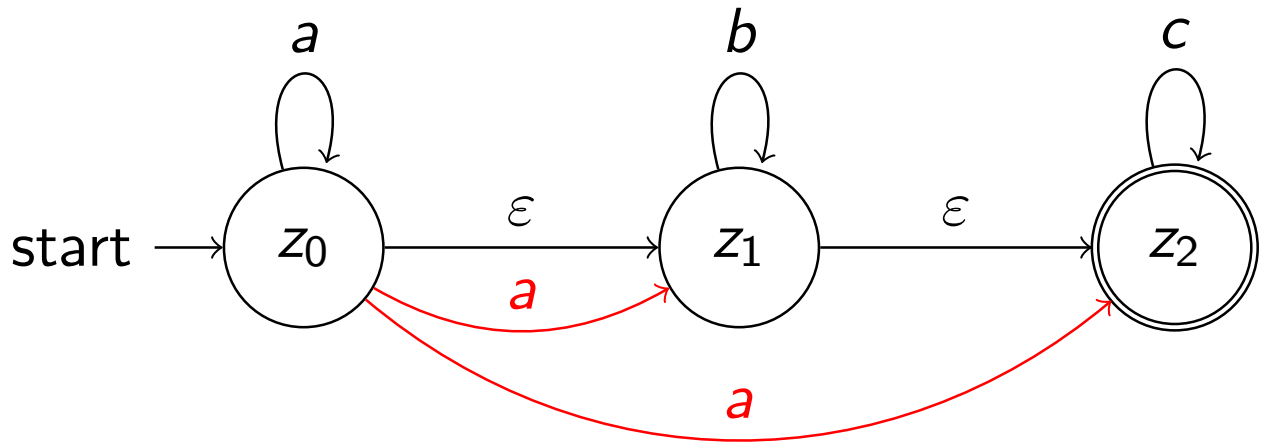

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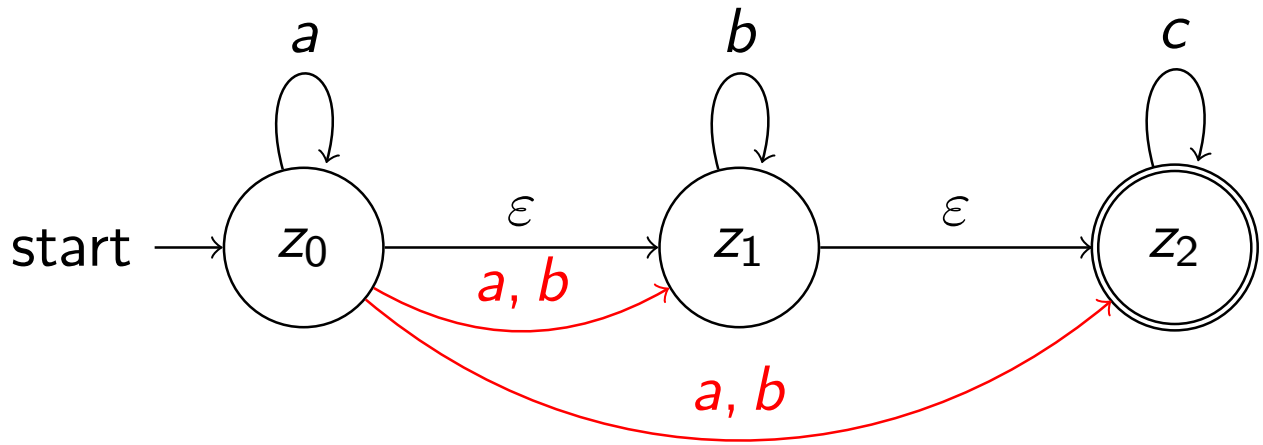

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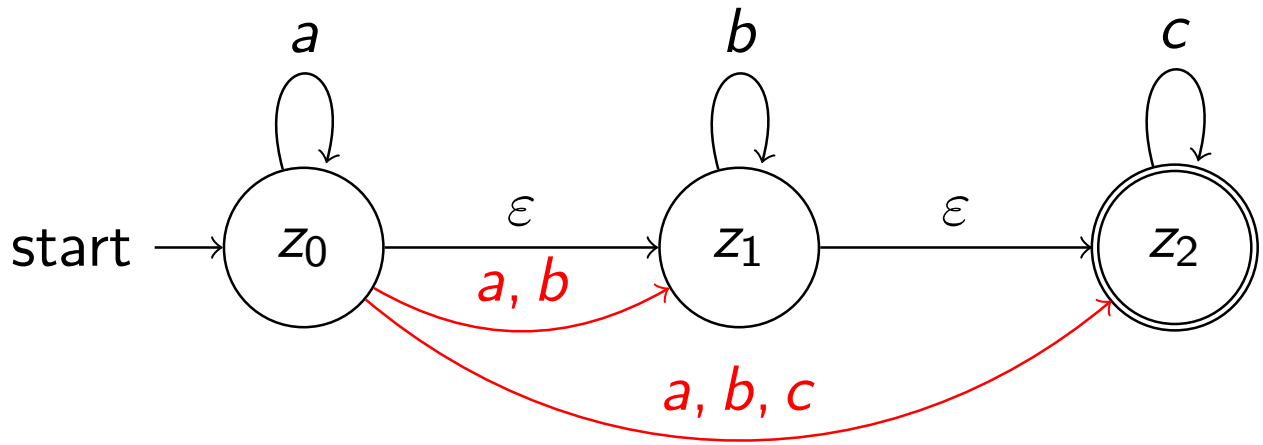

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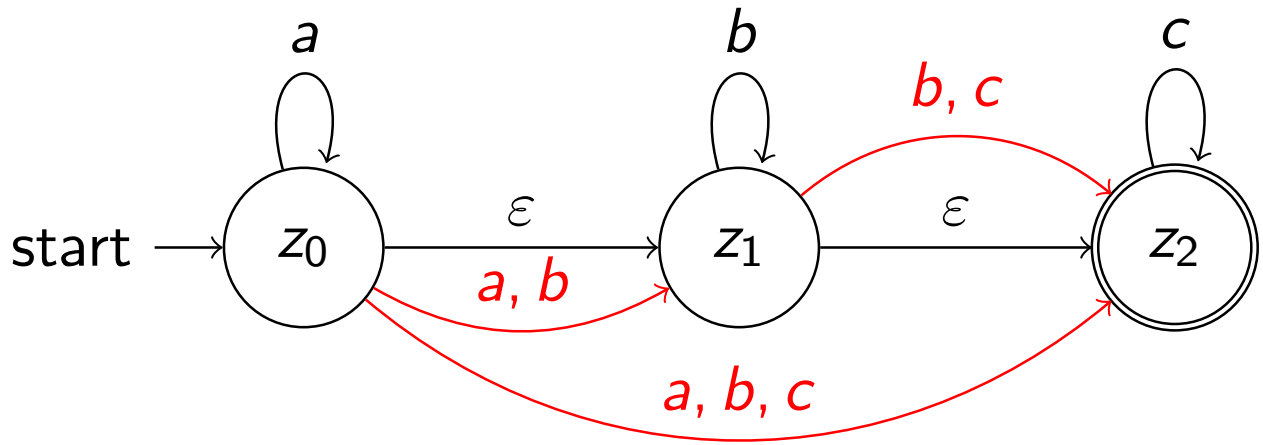

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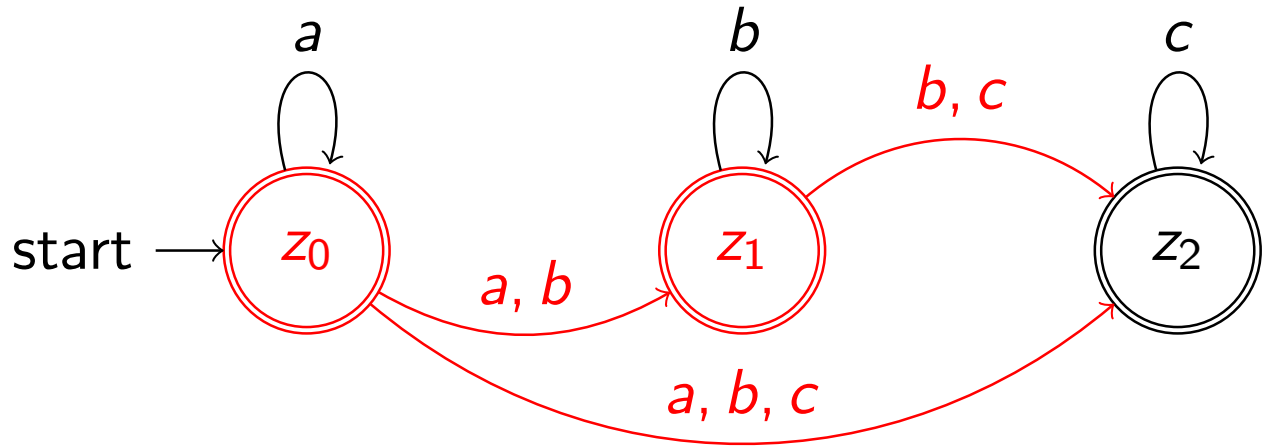

Regular Languages

Summary

The following are equivalent in the sense that they all produce/recognize regular languages:

- Regular Expressions
- Regular Grammars
- DFAs
- NFAs
- ε-FAs

State Minimization

Equivalence of Automata

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The minimal automaton is unique (despite trivial modifications such as renaming states).

Testing Equivalence of States

Equivalent States

Given a DFA, we say that states p and q are equivalent if for all input strings w:

$$\hat{\delta}(p, w) \text{ is an accepting state} \iff \hat{\delta}(q, w) \text{ is an accepting state}$$

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Remark

It's not required that $\hat{\delta}(p, w)$ and $\hat{\delta}(q, w)$ to be the same state; it suffices for either both to be accepting or both to be non-accepting.

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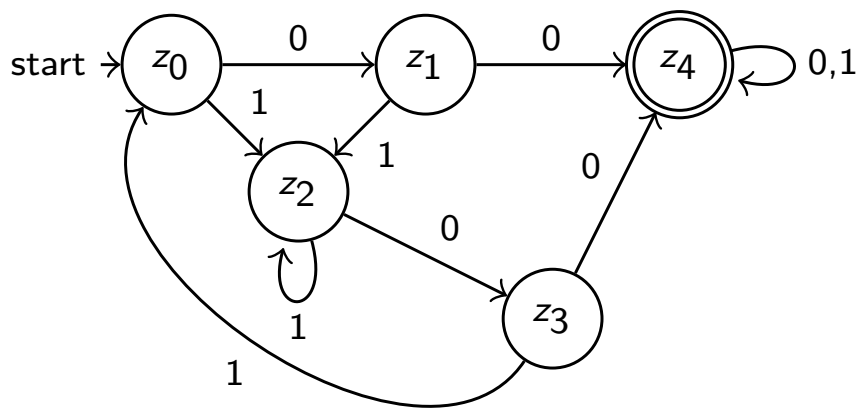

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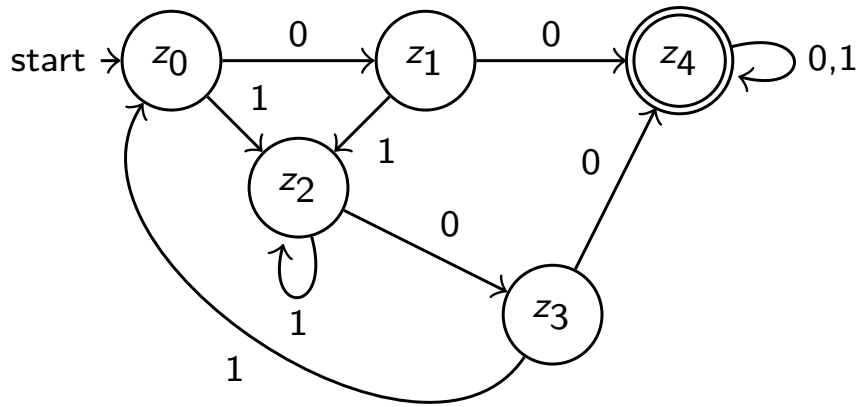

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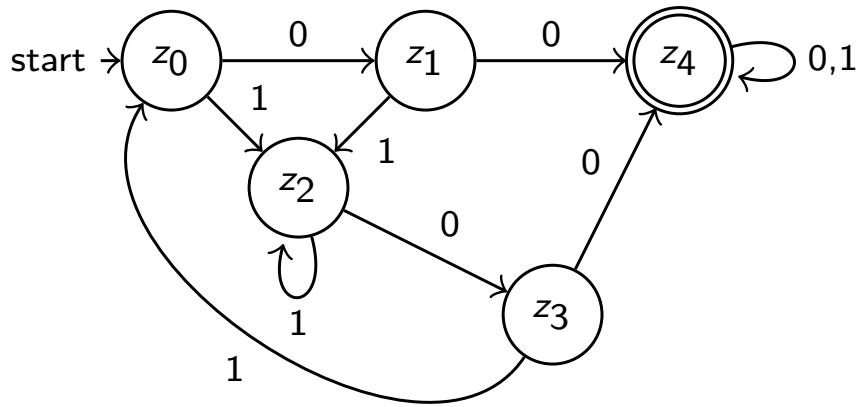

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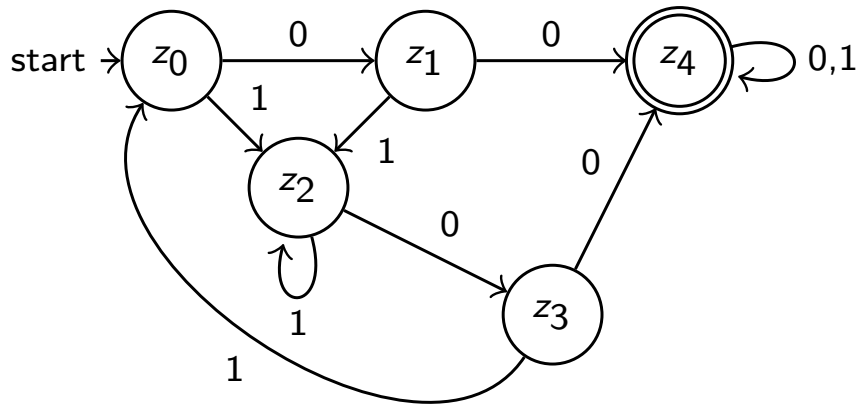

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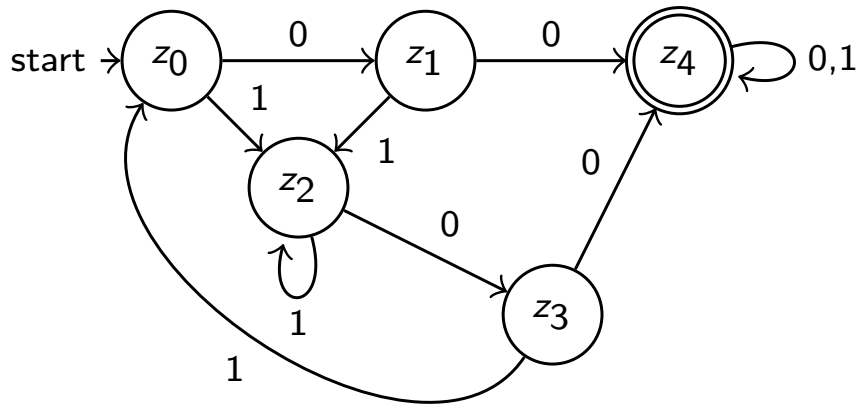

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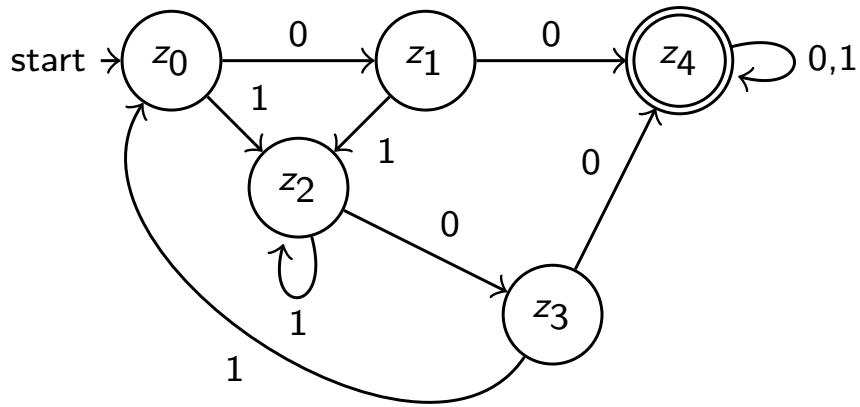

From these it follows that for every string w, $\hat{\delta}(z_1, w) = z_4 \Leftrightarrow \hat{\delta}(z_3, w) = z_4$. Why?

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2. Create an empty table of all state pairs $\{z, z'\}$ where $z, z' \in Z$ and $z \neq z'$.

3. Tag all pairs $\{z, z'\}$ where $z \in E$ and $z' \notin E$.

4. For each untagged pairs $\{z, z'\}$ and each $a \in \Sigma$ check if

$$\{\delta(z, a), \delta(z', a)\}$$

is already tagged. If yes, tag $\{z, z'\}$. Repeat this until nothing changes anymore.

5. Merge all untagged pairs of states.

State Minimization Algorithm

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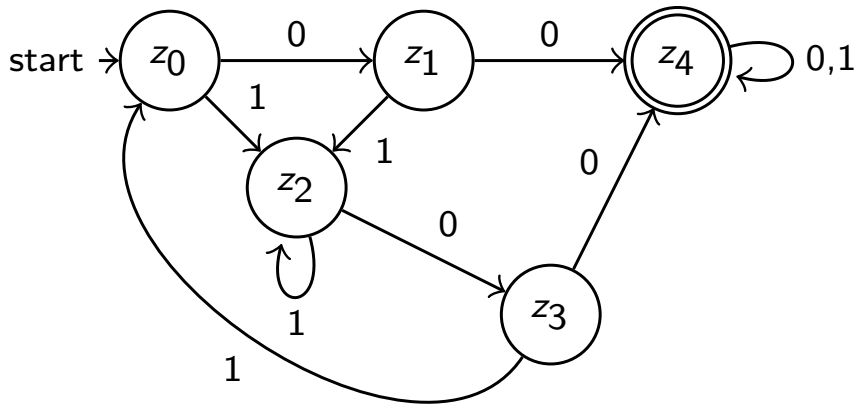
Output: the minimal automaton M'

State Minimization Algorithm

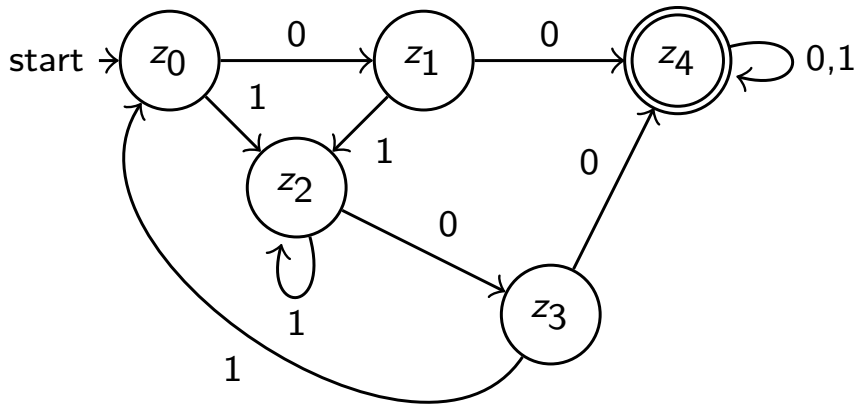
Example



State Minimization Algorithm

Example

Step 1: skip



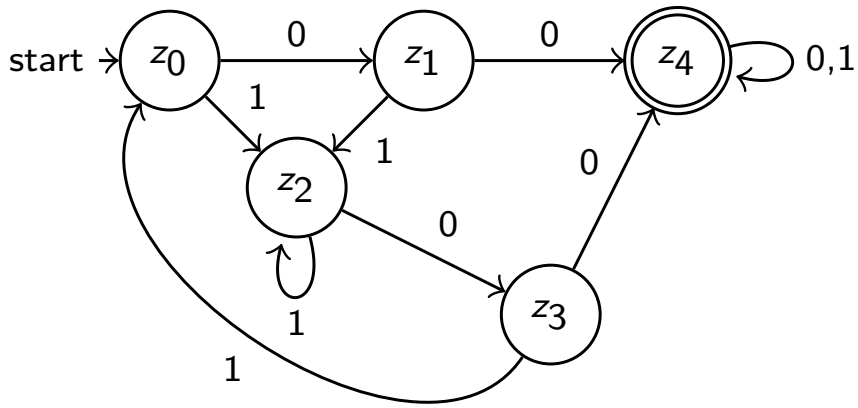
State Minimization Algorithm

Example

Step 1: skip

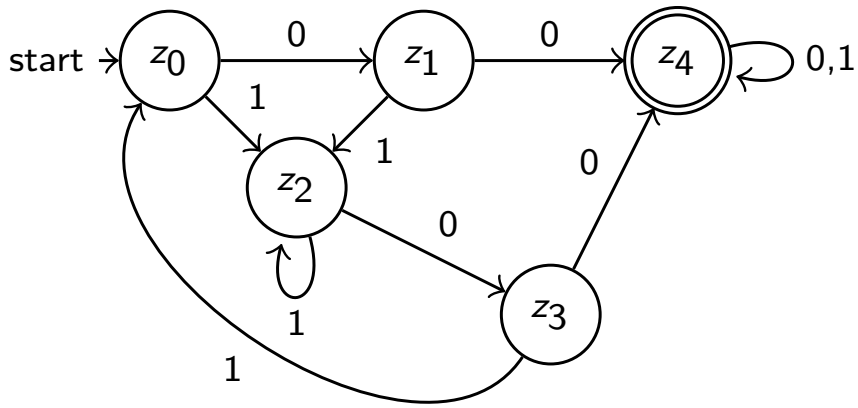
Step 2:

	z_0	z_1	z_2	z_3
z_4				
z_3				
z_2				
z_1				



State Minimization Algorithm

Example



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Step 2:

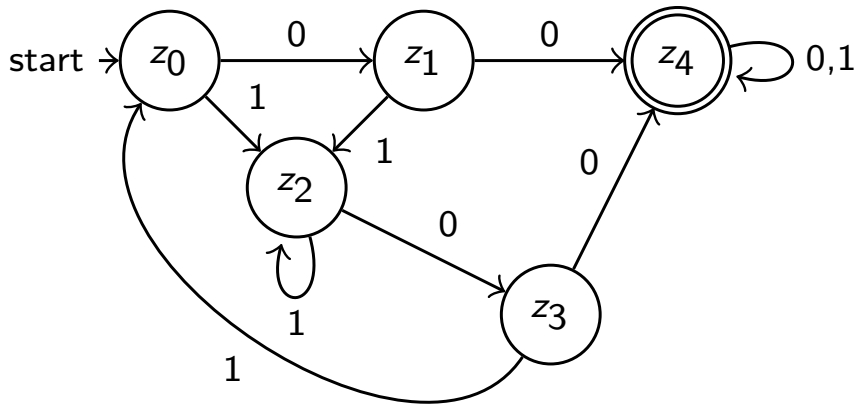
	z_0	z_1	z_2	z_3
z_4				
z_3				
z_2				
z_1				

Step 3:

	z_0	z_1	z_2	z_3
z_4	X	X	X	X
z_3				
z_2				
z_1				

State Minimization Algorithm

Example



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	z_0	z_1	z_2	z_3
z_4				
z_3				
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z_1				

Step 3:

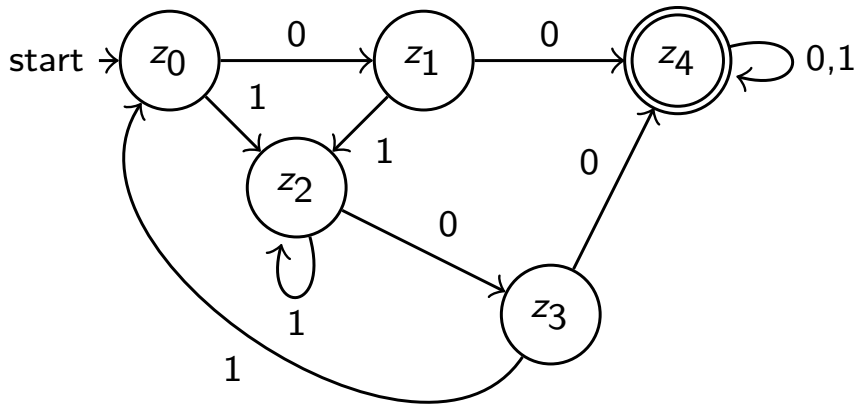
	z_0	z_1	z_2	z_3
z_4	X	X	X	X
z_3				
z_2				
z_1				

Step 4-5:

	z_0	z_1	z_2	z_3
z_4	X	X	X	X
z_3	X		X	
z_2		X		
z_1	X			

State Minimization Algorithm

Example



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z_3				
z_2				
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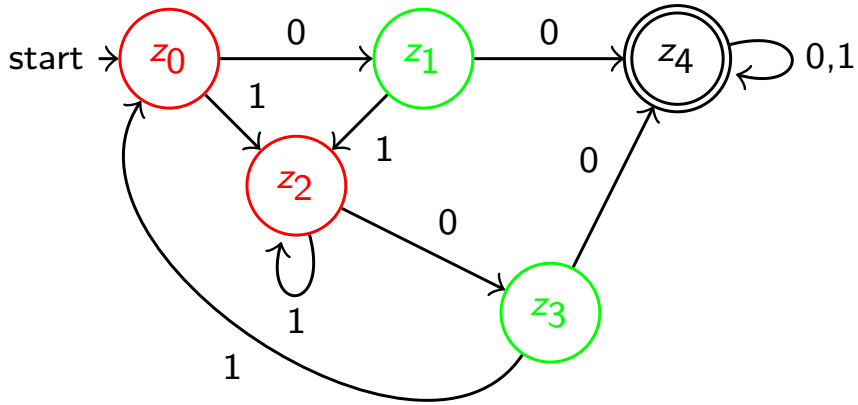
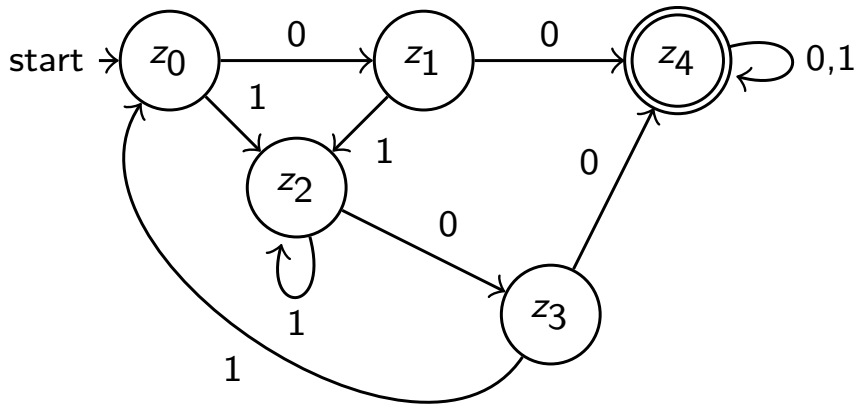
z_2		X		
z_1	X			

Step 6: The pairs z_0, z_2 and z_1, z_3 are equivalent and can be merged.



State Minimization Algorithm

Example



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z_3				
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z_2		X		
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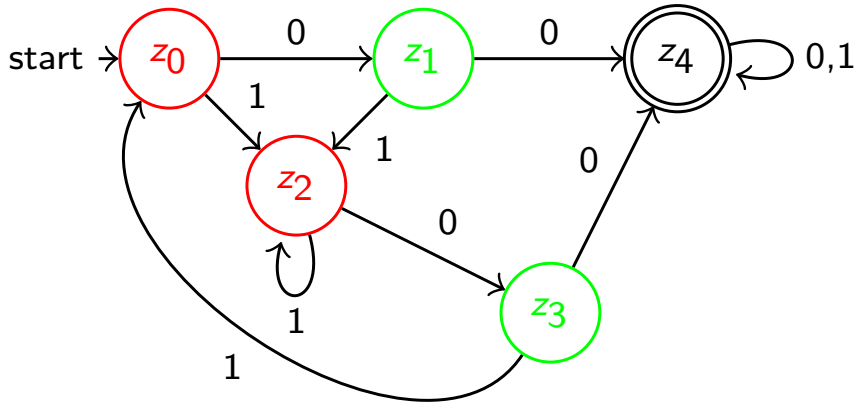
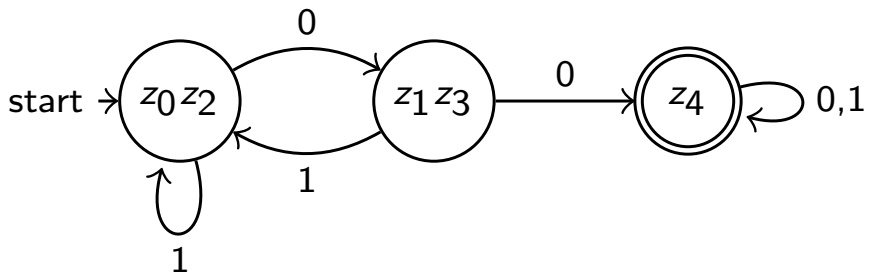


Output:



Closure Properties

Theorem

The regular languages are closed under the following operations

- union $(O \cup K)$
- intersection $(O \cap K)$
- complement $(\overline{O})$
- product (OK)
- star (O^*)

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Proof

The closure under union, product, and star has already been proved. We will now prove it for complement and intersection.

Closure Properties

Complement:

Let $M = (Z, \Sigma, \delta, z_0, E)$ be an automaton with $O = L(M)$ and a total transition function.

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Intersection:

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Intersection:

Due to the rules of de Morgan we have

$$O \cap K = \overline{\overline{O} \cup \overline{K}}.$$

Since the regular languages are closed under union and complement, they are thus also closed under intersection.

Intersection - alternative method

Product Construction

Let $M_1 = (Z, \Sigma, \delta_1, z_0, E)$ and $M_2 = (Q, \Sigma, \delta_2, q_0, F)$ be DFAs recognizing L_1 and L_2, respectively.

Intersection - alternative method

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Let $M_1 = (Z, \Sigma, \delta_1, z_0, E)$ and $M_2 = (Q, \Sigma, \delta_2, q_0, F)$ be DFAs recognizing L_1 and L_2, respectively.

Consider

$$M = (Z \times Q, \Sigma, \delta, (z_0, q_0), E \times F)$$

with

$$\delta((z, q), a) = (\delta_1(z, a), \delta_2(q, a)).$$

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Then M recognizes $L_1 \cap L_2$ (prove this as an exercise).

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Construct a DFA recognozing the language L of all binary strings with an odd number of 0's and odd number of 1's.

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Observe that $L = L_0 \cap L_1$ with

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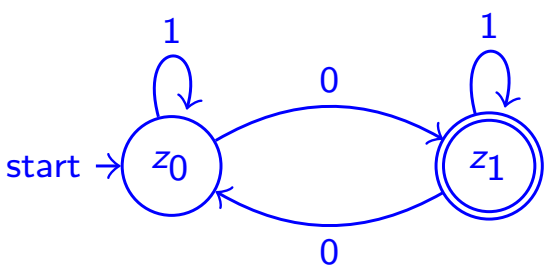

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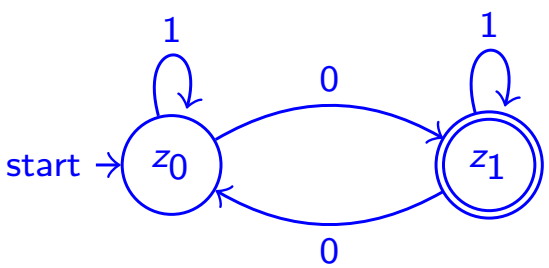

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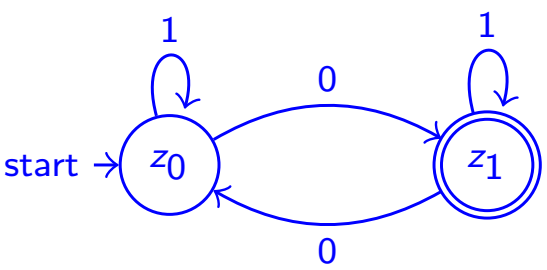

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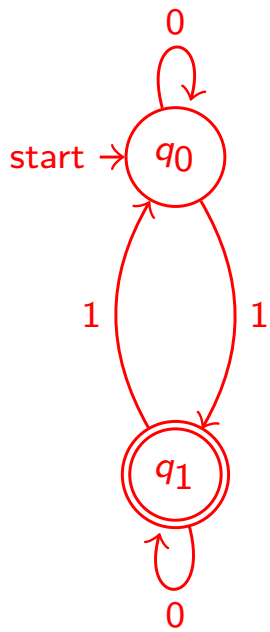

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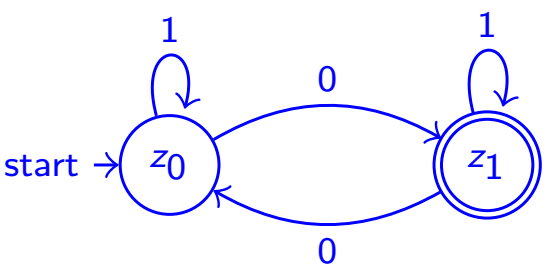

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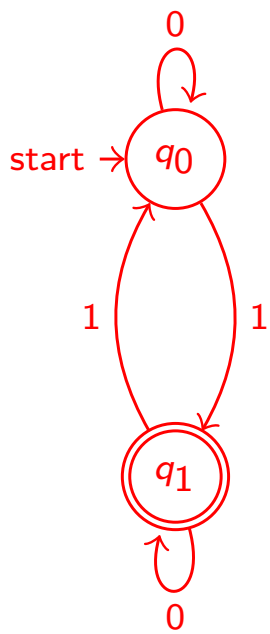

recognizes L_1

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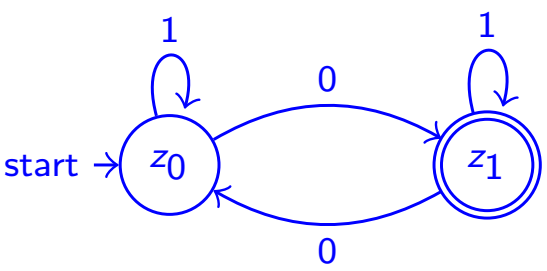
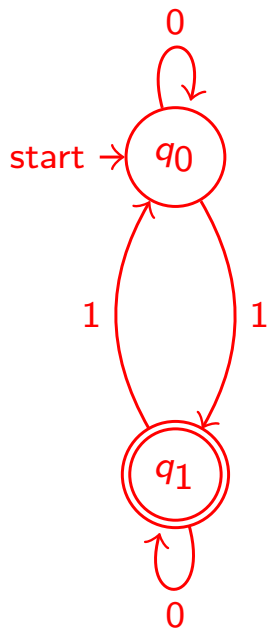


recognizes L_0

$\times$



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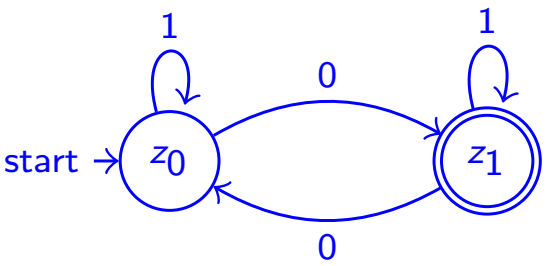
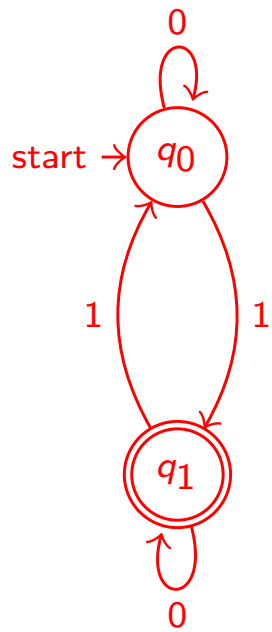


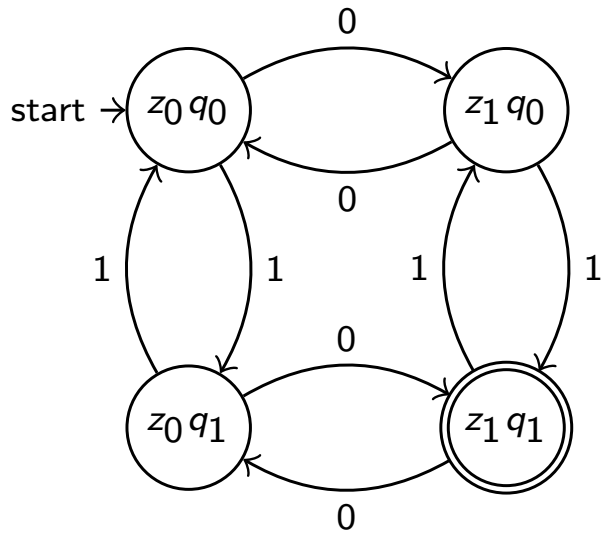
recognizes L_0

$\times$



recognizes L_1

$=$



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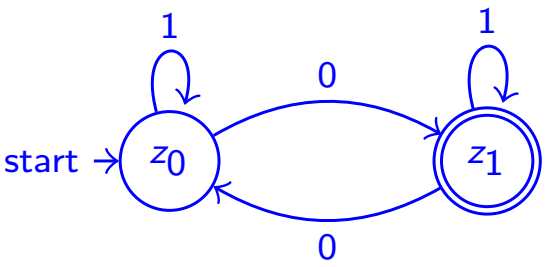
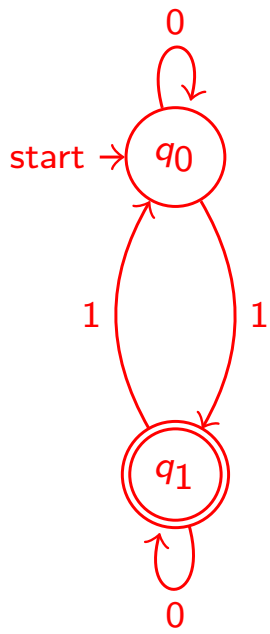


recognizes L_0

$\times$



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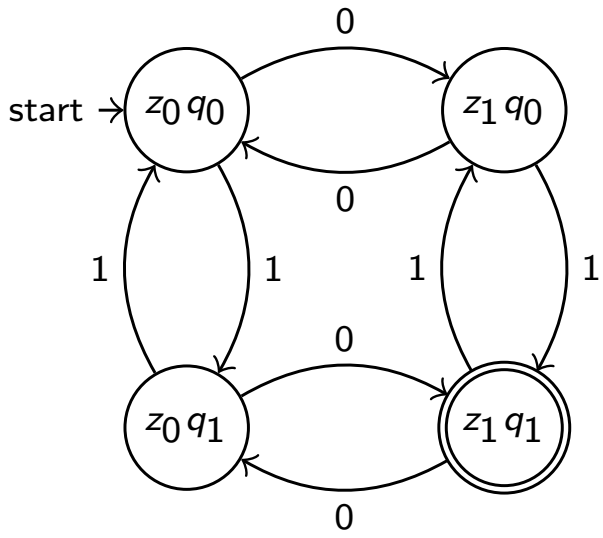

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