Automata Theory and Formal Languages

Summer Term 2026

10$^{\text{th}}$ Lecture

Context Free Languages: Closure Properties

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Context Free Languages: Closure Properties

Closure Properties

Theorem

The context-free languages are *closed* under the following operations

- union $(O \cup K)$
- product (OK)
- star (O^*)

Closure Properties

Theorem

The context-free languages are *closed* under the following operations

- union $(O \cup K)$
- product (OK)
- star (O^*)

They are not closed under

- intersection $(O \cap K)$
- complement $(\overline{O})$

Closure Properties

Union

Let $G_1 = (V_1, \Sigma_1, P_1, S_1)$ and $G_2 = (V_2, \Sigma_2, P_2, S_2)$ be two context free grammars with $V_1 \cap V_2 = \emptyset$ then

$$G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, P_1 \cup P_2 \cup \{S \to S_1 \mid S_2\}, S)$$

satisfies $L(G) = L(G_1) \cup L(G_2)$. The newly introduced rule is context free and thus G is also context free.

Closure Properties

Product

Let $G_1 = (V_1, \Sigma_1, P_1, S_1)$ and $G_2 = (V_2, \Sigma_2, P_2, S_2)$ be two context free grammars with $V_1 \cap V_2 = \emptyset$ then

$$G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, P_1 \cup P_2 \cup \{S \to S_1 S_2\}, S)$$

satisfies $L(G) = L(G_1)L(G_2)$. The newly introduced rule is context free and thus G is also context free.

Closure Properties

Kleene Star

Let $G_1 = (V_1, \Sigma, P_1, S_1)$ be a context free grammar where S_1 does not occur on any right hand side of the production (without loss of generality) then

$$G = (V_1 \cup \{S\}, \Sigma, P_1 \cup \{S \to \varepsilon, S \to S_1, S_1 \to S_1 S_1\} \setminus \{S_1 \to \varepsilon\}, S)$$

satisfies $L(G) = L(G_1)^*$. The newly introduced rules are context free and thus G is also context free.

Closure Properties

Intersection

We consider the languages

$$L_1 = \{a^i b^j c^j \mid i, j > 0\}$$
$$L_2 = \{a^i b^i c^j \mid i, j > 0\}$$

are both context free.

Closure Properties

Intersection

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are both context free.

An exemplary grammar for L_1 is given by

$$S \to AB,$$
$$A \to a \mid aA$$
$$B \to bc \mid bBc$$

Closure Properties

Intersection

The intersection of L_1 and L_2 is given by

$$L_1 \cap L_2 = \{a^i b^i c^i \mid i > 0\}$$

is, however, not context free. (without proof, $\rightarrow$ Pumping lemma for context free languages $\rightarrow$ Schöning). Thus, the context-free language are not closed under intersection

Closure Properties

Complement

One of the rules of de Morgan is

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}.$$

If the context free languages would be closed under complement, they thus would also be closed under intersections. Since this is not the case, the context-free language cannot be closed under complement.