Automata Theory and Formal Languages

Summer Term 2026

17th Lecture

Multi-Tape and Universal Turing Machines

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Turing Machine

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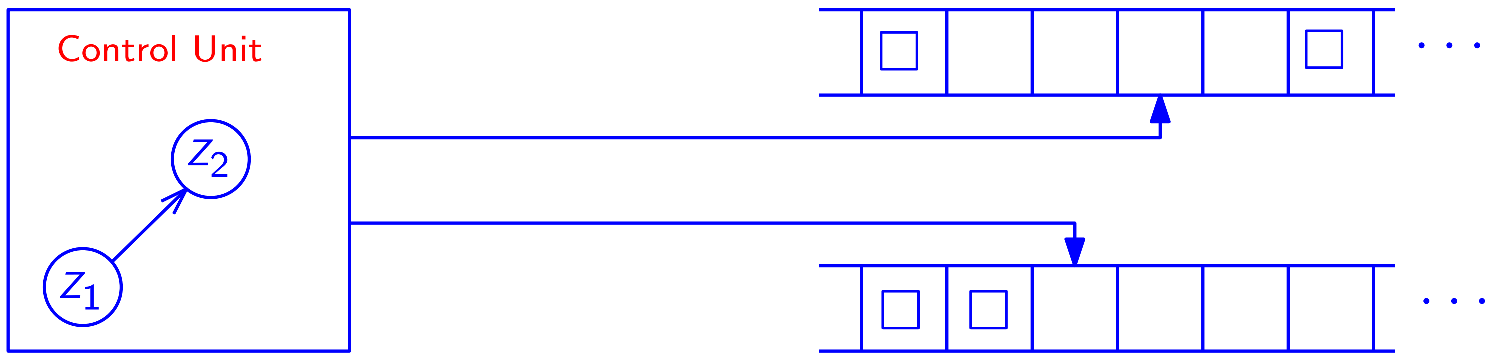
k-tape TM:

$$\delta : Z \times \Gamma^k \to Z \times \Gamma^k \times \{L, R, N\}^k$$

$$\delta(z, a_1, \ldots, a_k) = (z', b_1, \ldots, b_k, x_1, \ldots x_k)$$



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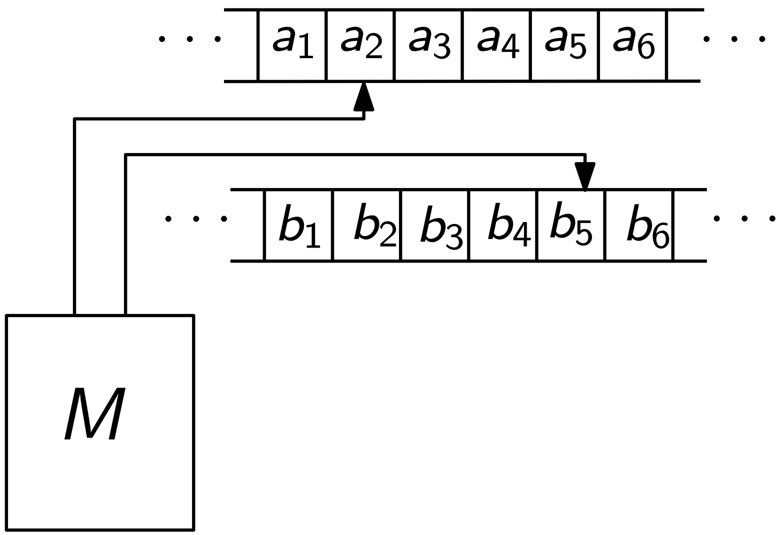

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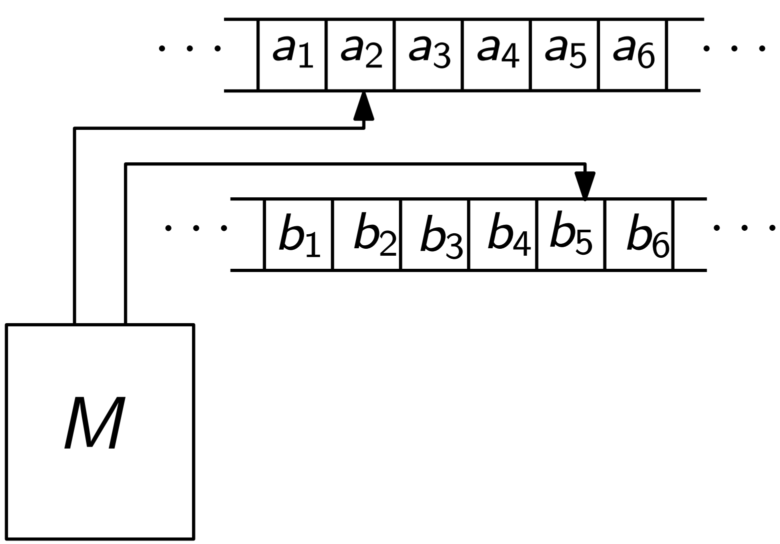

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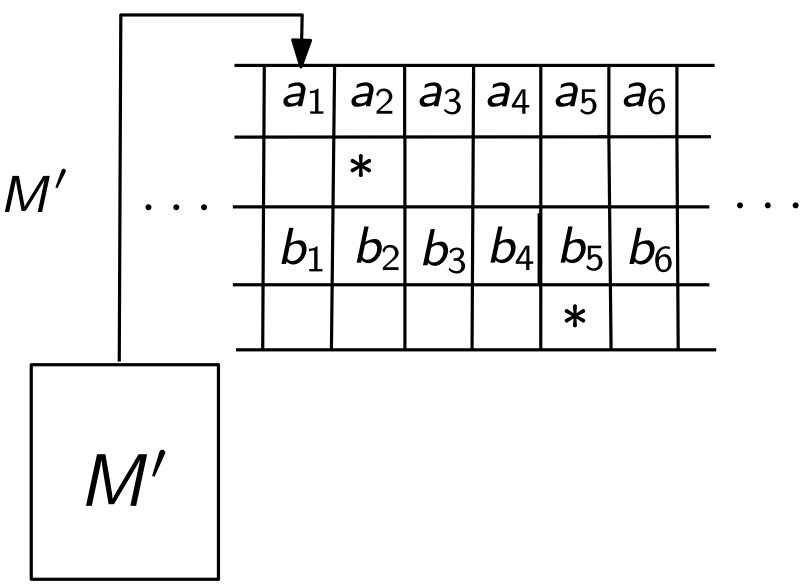

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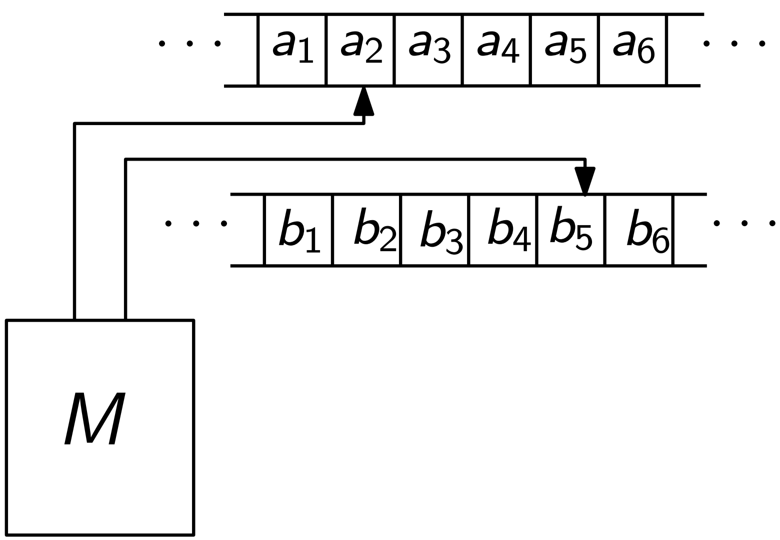

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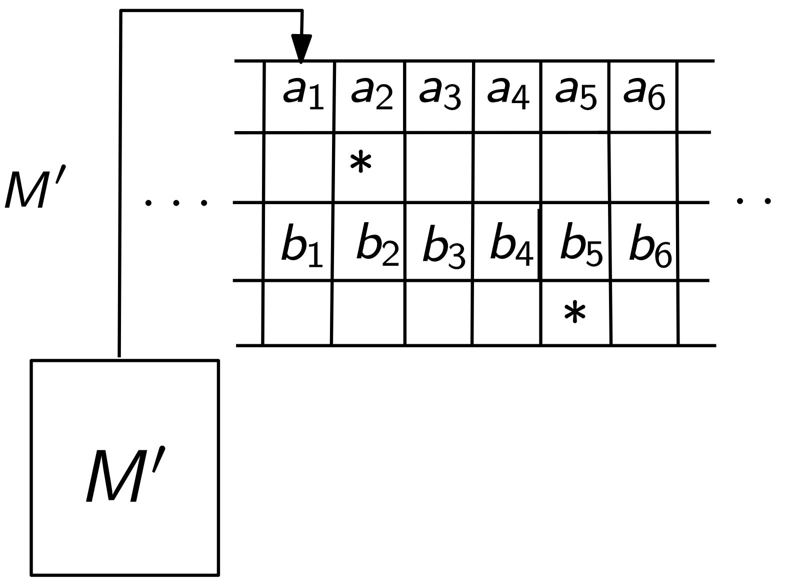

Exercise
2-tape Turing machine for $a^n b^n c^n$ that requires only a single pass.

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Example Input: Binary number $x = \{0, 1\}^*$
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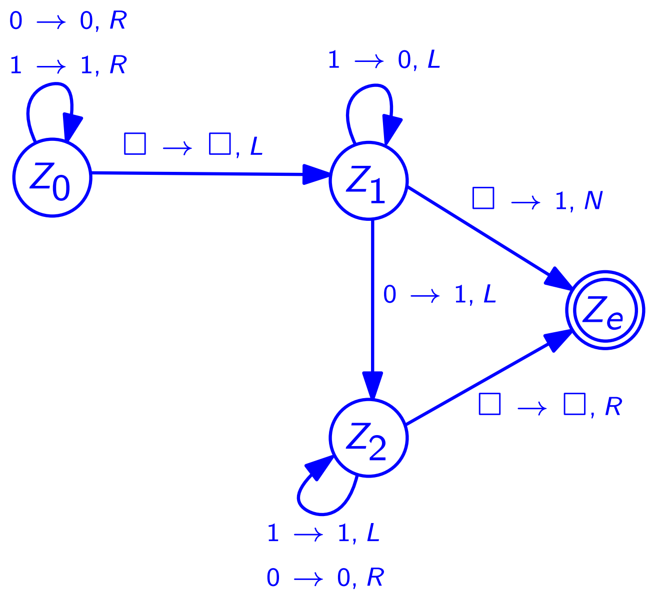

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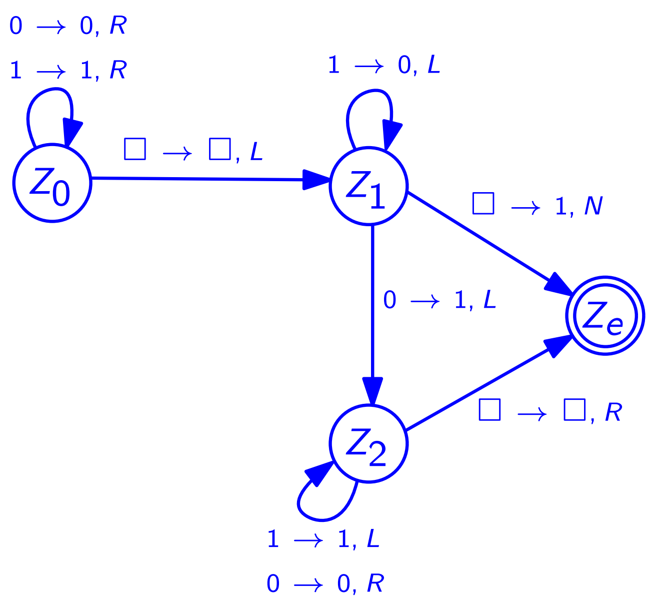

(i) $x = 101$

head position

State	Tape
z_0	$\overline{1}01$
z_0	$1\overline{0}1$
z_0	$10\overline{1}$
z_0	$101\overline{\square}$
z_1	$10\overline{1}$
z_1	$1\overline{0}0$
z_2	$\overline{1}10$
z_2	$\overline{\square}110$
z_e	$\overline{1}01$

$z_0 101 \vdash 1 z_0 01 \vdash 10 z_0 1 \vdash$
$101 z_0 \square \vdash 10 z_1 1 \vdash 1 z_1 00 \vdash$
$z_2 110 \vdash z_2 \square 110 \vdash z_e 110$

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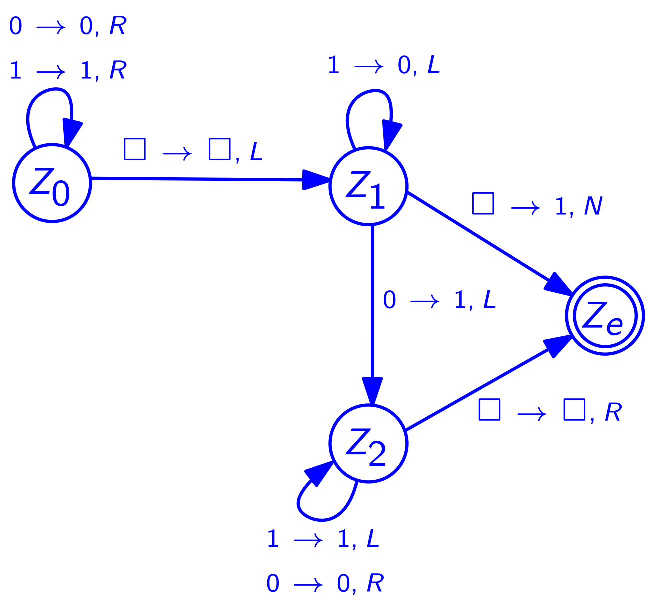

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z_2	$\overline{1}10$
z_2	$\overline{\square}110$
z_e	$\overline{1}01$

$z_0101 \vdash 1z_001 \vdash 10z_01 \vdash$
$101z_0\square \vdash 10z_11 \vdash 1z_100 \vdash$
$z_2110 \vdash z_2\square110 \vdash z_e110$

(ii) $x = 11$

State	Tape
z_0	$\overline{1}1$
z_0	$1\overline{1}$
z_0	$11\overline{\square}$
z_1	$1\overline{1}$
z_1	$\overline{1}0$
z_1	$\overline{\square}00$
z_e	$\overline{1}00$

$z_011 \vdash 1z_01 \vdash 11z_0\square \vdash$
$1z_11 \vdash z_110 \vdash z_1\square00 \vdash$
z_e100

Universal TM

The program of a turing machine is realized *in hardware*. This is like building the circuits of a CPU for a specific assembler program

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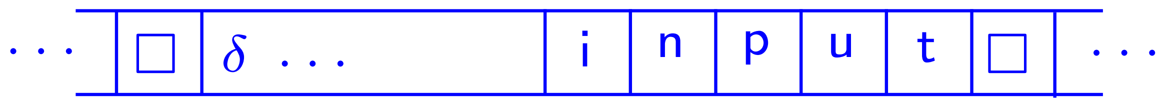
Idea

Put the δ description on the tape followed by the input



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The UTM can be used to prove the undecidability of the halting problem.

$$H = \{ < M > x \mid M \text{ is a deterministic turing machine that terminates on input } x \}$$

Example Type 1 Language that is not Type 2

The following language is a type 1 language, which is not type 2

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Grammar:

$$S \xrightarrow{(1)} LaR$$

$$L \xrightarrow{(2)/(3)} LD \mid \varepsilon$$

$$Da \xrightarrow{(4)} aaD$$

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$$R \xrightarrow{(6)} \varepsilon$$

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Example derivation:

$$S \overset{(1)}{\Rightarrow} LaR \overset{(2)}{\Rightarrow}{}^3 LDDDaR \overset{(3)}{\Rightarrow} DDDaR$$

$$\overset{(4)}{\Rightarrow} DDaaDR \overset{(4)}{\Rightarrow} DaaDaDR \overset{(4)}{\Rightarrow} aaDaDaDR$$

$$\overset{(4)}{\Rightarrow} aaaaDDaDR \overset{(4)}{\Rightarrow}{}^* aaaaaaaaDDDR$$

$$\overset{(5)}{\Rightarrow} aaaaaaaaDDR \overset{(5)}{\Rightarrow}{}^* aaaaaaaaDR$$

$$\overset{(5)}{\Rightarrow} aaaaaaaaR \overset{(6)}{\Rightarrow}{}^* aaaaaaaa = a^8 = a^{2^3}$$