Automata Theory and Formal Languages

Summer Term 2026

19th Lecture

The Busy Beaver Problem

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- *M(n)*: set of Turing machines with *n* states w/ $\Sigma = \{0,1\}$

Based on work of Bram van Heuveln

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Busy Beaver Problem:

Find $\Sigma(n) = \max\{\Sigma(M) | M \in M(n)\}$

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Let's stick to binary alphabet and a standard configuration.
But all proofs can be modified to accommodate all other types.

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Proof by Contradiction: Suppose there is some Turing-Machine *Σ* that computes *Σ(n).*
Then consider the following Turing-machine Q, where k is the number of states of the last 3 components:

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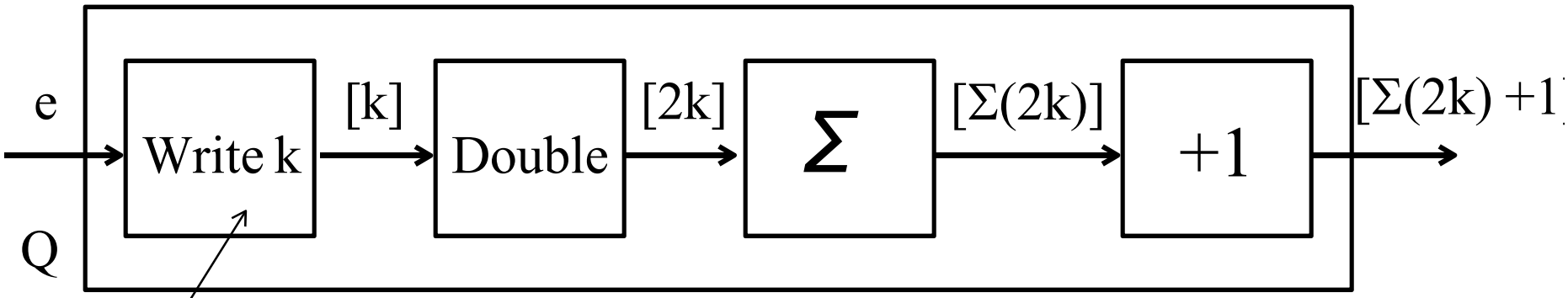

Can be implemented using k states. So, Q has $2k$ states … and outputs $\Sigma(2k) + 1$ when started on an empty tape – contradiction.

The Shifting Problem

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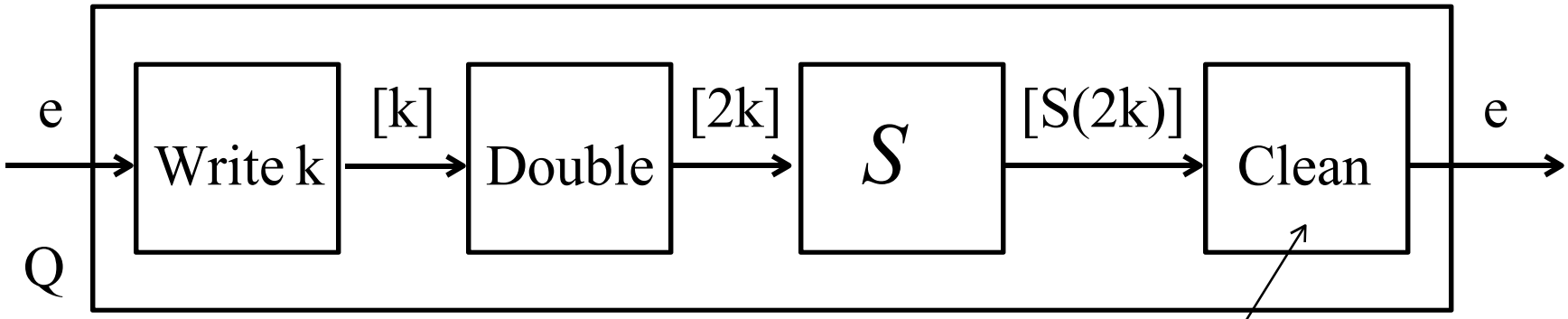

Cleaning the tape will take at least $S(2k)$ steps. So Q has $2k$ states but takes more than $S(2k)$ steps before halting – contradiction.

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If a Turing machine is still running after *S(n)* steps, you know it is a non-halter. For all the others *Σ(M)* can be determined, and now simply take the max.

Computability, uncomputability, and the Church-Turing Thesis

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- So, the proof appeals to the Church-Turing Thesis, which states that anything that is computable is Turing-computable (so: since we know that *Σ(n)* is Turing-uncomputable, *Σ(n)* is not computable).

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In many of the other proofs we make similar use of the Church-Turing Thesis, i.e. from now on by computability we mean Turing-computability.

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- So, if problem A reduces to problem B, then if we know that problem A cannot be solved, then we know that B cannot be solved either.
- In the proof, we reduced the computability of $\Sigma(n)$ to the computability of $S(n)$.

The General Busy Beaver Problem

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- For any particular problem you can show this either by a direct proof, or by reducing it into another problem that you already established to be uncomputable.

Upper Bounds

Another interesting thing to note about the proof (from 4 slides ago) is that it uses the technique of using upper bounds: if we know an upper bound to the number of steps a machine can take before halting (such as $S(n)!$), then we can determine any machine with n states to be a halter or non-halter simply by running it: any machine that is still running after $S(n)$ steps is a non-halter.

So, once you have discarded all non-halters, you can simply run all halters to completion to figure out any kind of Busy Beaver function you want.

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  … but that would be a mistake!
  - Remember that $S(n)$ gives an upper-bound to the number of steps taken by all machines with n states … when started on an empty tape!
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Still, the two problems are intimately related, but more complex.

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The other way is more difficult.

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Question: does the unsolvability of the Halting problem imply the unsolvability of ETHP?

Uncomputability of the Empty Tape Halting Problem

Yes! First we devise the Create-M_T routine that, given any M and T, constructs a machine M_T that, when given an empty tape, first puts T on the tape, and then runs M on that tape.

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Now assume the Empty Tape Halting Problem is solvable. Then the Halting Problem is solvable as well:

Empty Tape Halting Problem and $S(n)$

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Proof:

- If ETHP is solvable, then we can figure out $S(n)$ by discarding all non-halters, running all others to completion, and determine max.
- If $S(n)$ is computable, then ETHP is solvable: simply start running any machine on an empty tape, and if it is still running after $S(n)$ steps, then it is a non-halter, otherwise it is a halter.

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  - E.g. for the quintuple formalization, you can show that if M_1 has n states, then such a M_2 can be constructed with 20n states.

$S(n)$ and $\Sigma(n)$

Suppose $S(M_1) = S(n)$ (i.e. M_1 is a machine with n states that makes the most steps for any machine with n states) we thus have $S(n) \leq \Sigma(f(n))$ for some computable $f(n)$.

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So, if $\Sigma(n)$ is computable, then $S(n)$ becomes computable too: simply start any machine with n states on an empty tape, and any machine that still runs after having taken $\Sigma(f(n))$ steps must be a non-halter. For all halters, simply determine the maximum number of steps taken. In short, $\Sigma(f(n)$ provides an upper-bound for $S(n)$.

Summing it all up: Busy Beaver and Halting Problem

We have shown that:

- The Halting Problem is solvable if and only if the Empty Tape Haling Problem is solvable.
- The Empty Tape Halting problem is solvable if and only if $S(n)$ is computable.
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- So, these are all equivalent statements.
- In particular, all these problems (functions) are unsolvable (uncomputable).

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Then $S(n)$ would be computable too: all machines that still run after f(n) are non-halters, so $S(n)$ can be determined by examining all halters.

As $S(n)$ is not computable, we know that no such function exists: there is no computable upper bound for $S(n)$.

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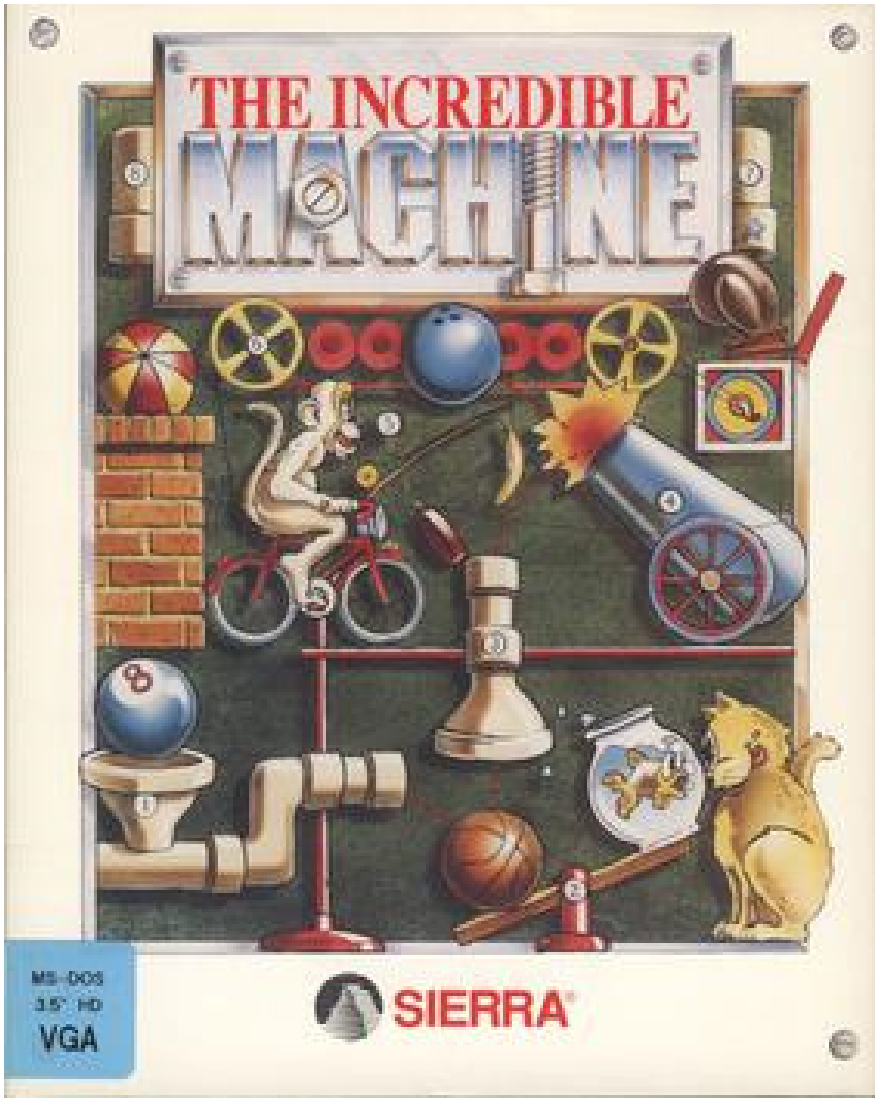
- And, you can easily come up with some computable functions that grow crazy fast.
- But $S(n)$ will grow even faster than that!
- Moreover, since we found that $S(n) \leq \Sigma(f(n))$ for some computable $f(n)$, we know that $\Sigma(n)$ also grows faster than any computable function. (otherwise, we would once again have a computable upper bound for $S(n)$).

Established values for $\Sigma(n)$ (for quintuple TM's)

n	$\Sigma(n)$
1	1 (trivial)
2	4 (easy)
3	6 (Lin and Rado)
4	13 (Brady)
5	>= 4098 (Marxen et al.)
6	> $3.514 * 10^{18276}$ (Marxen et al.)
7	Huge
8	Insane

Why bother finding $\Sigma(n)$?



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Goldbach's Conjecture: Write a program A that sequentially tests every even number ≥ 4 sequentially if it is the sum of two prime numbers or not. Simulate A on an n-state Turing machine. If it finds a counterexample, it halts and indicates that. But if the conjecture is true, then A will never halt.

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So if we know $S(n)$ we can decide (in finite time) if it will ever halt by simply running the machine that many steps. And if, after $S(n)$ steps, the machine does not halt, we know that it never will and so there are no counterexamples to Goldbach.

Problems in Determining $\Sigma(n)$

First, search space is fairly big:

- Explicit halting state: $|M(n)| = (4n + 4)^{2n}$
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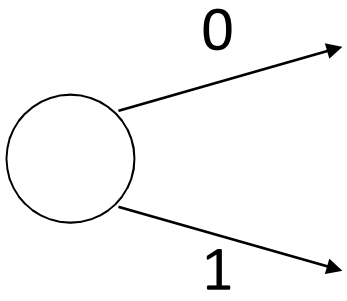
But more importantly, the behavior of even small machines is strange and hard to grasp, and it is hard to tell whether they halt or not.

Size of Search Space

For every state:



For every transition:

Quintuple:

0 or 1 and L or R

⇒ 2*2 = 4 possibilities

Quadruple:

0 or 1 or L or R

⇒ 4 possibilities

In all cases: 2n state-symbol pairs

Explicit: n + 1 possible new states with full transition

Implicit: n possible new states with full transition + empty transition to halting state

n	$(4n + 4)^{2n}$	$(4n + 1)^{2n}$
1	64	25
2	20,736	6,561
3	16,777,216	4,826,809
4	$\sim 2.56 * 10^{10}$	$> 10^{9}$
5	$> 10^{13}$	$> 10^{13}$
6	$> 10^{17}$	$> 10^{16}$

Strange Behavior

For $n = 5$ (quintuple), some machines only come to a halt after over 47 million steps (this is the suspected shifting champion … but we haven't been able to prove that yet)

- For $n = 6$, some machines halt after more than 10^{36534} steps.
- In 1983, Brady conjectured that it's impossible to classify all machines with n = 5
- In 2024, the case n = 5 was settled by graduate students!
- Still, it's unlikely we'll ever settle the n = 6 case.
- And it's a virtual certainty we'll never settle the n = 7 case!

Attacking the Busy Beaver Function

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This is done by:

– Reducing the search space

– Writing custom-made computer programs to simulate the behavior of Turing-machines and detecting knowable forms of non-halting behavior.

Reducing the Search Space

The search space can be reduced in a number of ways:

- Perform some initial analysis to rule out certain corners of the search space
- Only consider one machine of a set of machines that can be shown to have equal productivity
- Use Tree normalization method to create only 'relevant' machines.

Halters, Non-halters, and Holdouts

As the halting problem cannot be solved in general, any practical algorithm that tries to figure out halting behavior is such that for any machine M:

- A eventually declares that M halts (M is a halter)
- A eventually declares that M does not halt (M is a non- halter)
- A eventually declares that it doesn't know whether M halts or not (M is a *holdout*)

Taking care of the holdouts

To take care of the holdouts, do the following:

- Simulate a holdout for some number of steps
- Observe the behavior, and see if there is some pattern present which one can use to prove that the machine will halt or not
- Try and generalize the pattern of behavior, and incorporate it into the original algorithm to obtain a new and improved algorithm
- If there are still holdouts left with the new algorithm, repeat this routine

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- Lin and Rado reduced the 56871 holdouts from 1963 to 40 holdouts in 1965, and analyzed those last 40 by hand to determine $\Sigma(3)$.

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- Brady obtained 0 holdouts for n = 3, and 218 holdouts for n = 4. These "were examined by means of voluminous printouts of their histories along with some program extracted features. It was determined to the author's satisfaction that none of these machines will ever stop."