Automata Theory and Formal Languages

Summer Term 2025

22nd Lecture

Program Verification

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Verification

Example

An automaton for the reachable configurations of a program

```
while x = 1 do
    if y = 1 then
        x ← 0
    y ← 1 − x
end
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Verification

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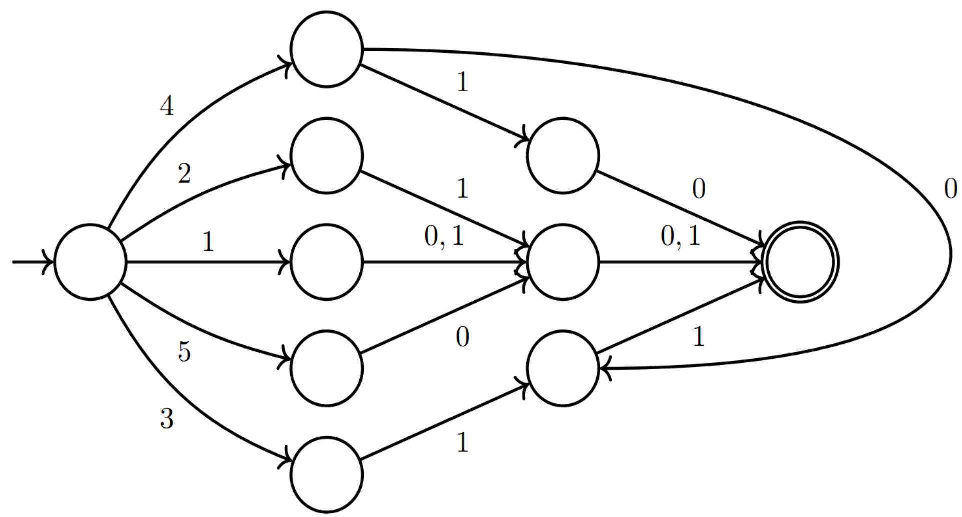

Based on Sections 8.1-8.4 of Esparza: Automata Theory - An algorithmic approach

Verification

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- We reduce the verification problem to language inclusion between implementation and specification

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- Configuration: triple $[\ell, n_x, n_y]$ where
  - ℓ is the current value of the program counter, and
  - n_x, n_y are the current values of x, y

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Examples: $[1, 1, 1], [5, 0, 1]$

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Verification

$$
\begin{array}{ll}
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2 & \quad \textbf{\textit{if}}\ y = 1\ \textbf{\textit{then}} \\
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- Potential execution: finite or infinite sequence of configurations

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Examples:

$[1, 1, 1][4, 1, 0]$

$[2, 1, 0][5, 1, 0]$

$[1, 1, 0][2, 1, 0][4, 1, 0][1, 1, 0]$

Verification

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while x = 1 do
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- Execution: potential execution starting at an initial configuration, and where configurations are followed by their “legal successors” according to the program semantics.

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$[1, 1, 0][2, 1, 0][4, 1, 0][1, 1, 0]$

- Full execution: execution that cannot be extended (either infinite or ending at a configuration without successors)

Verification as a language problem

- Implementation: set E of executions

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  - subset P of the potential executions that satisfy a property, or
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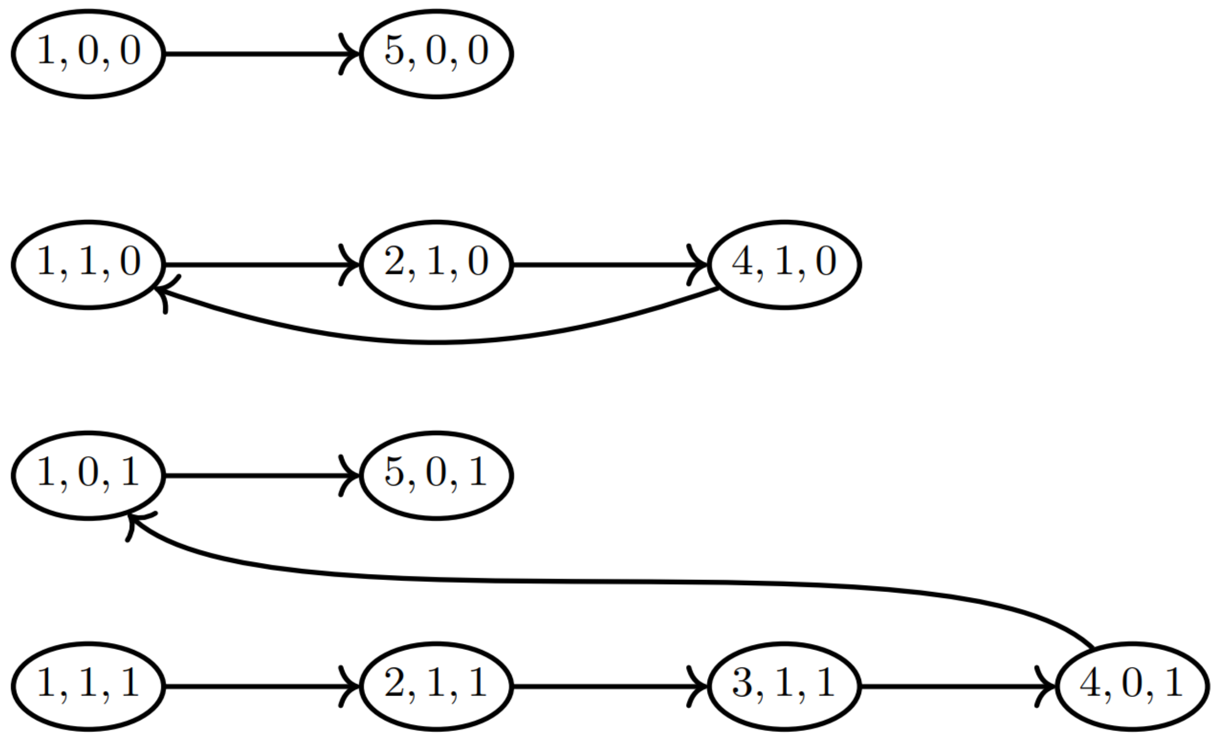
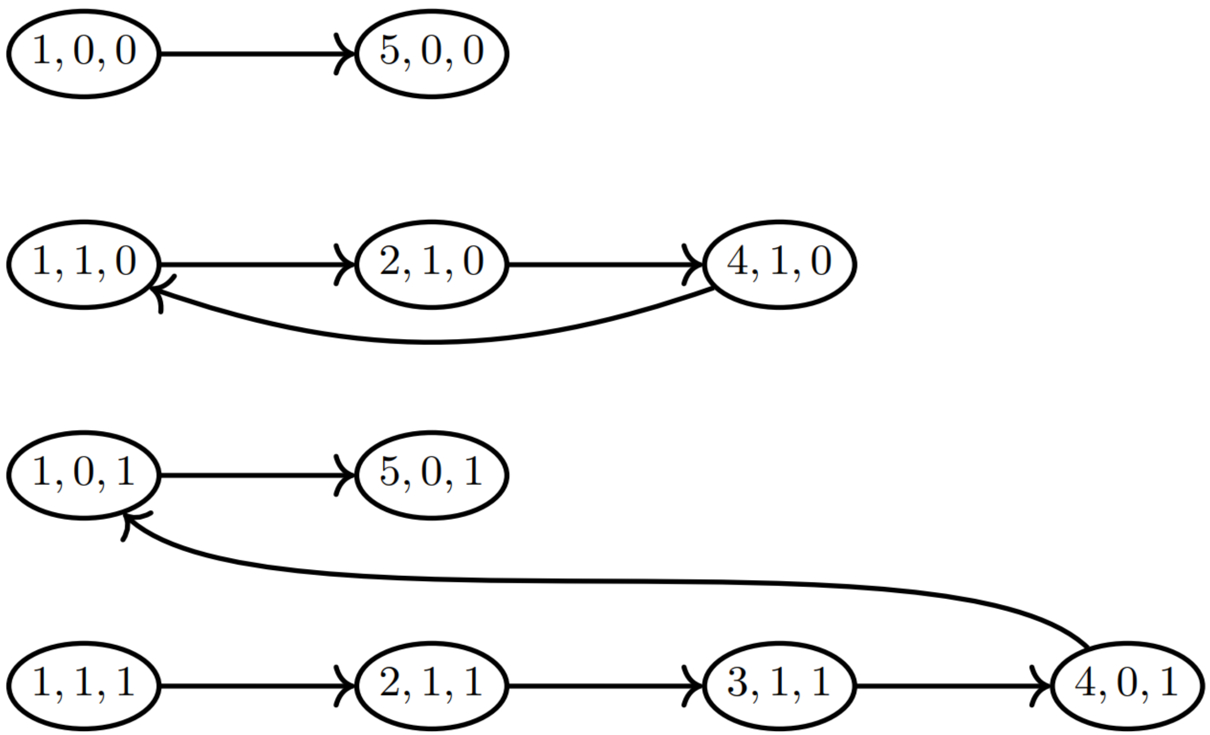
How often is the case that E, P, V are regular?

System NFA

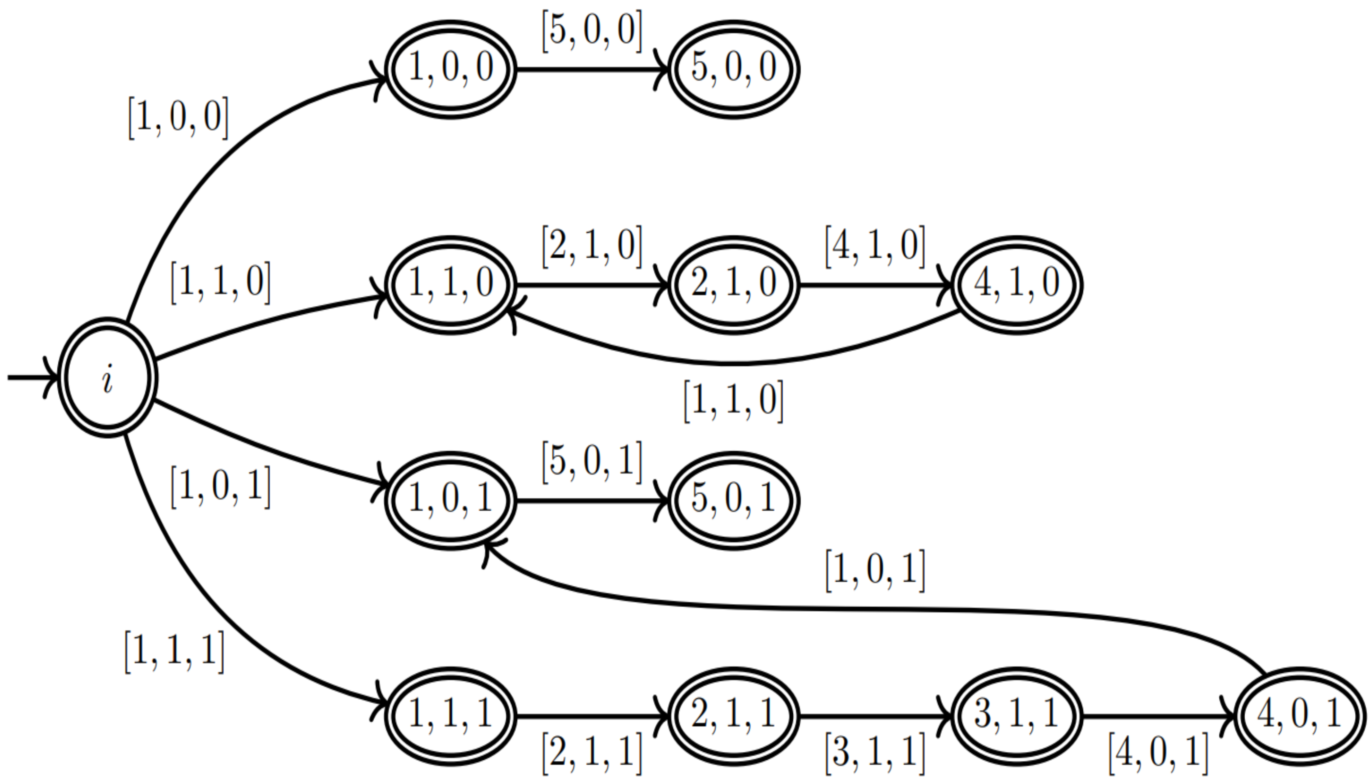
$\textbf{\textit{while}}\ x = 1\ \textbf{\textit{do}}$
$\quad \textbf{\textit{if}}\ y = 1\ \textbf{\textit{then}}$
$\qquad x \leftarrow 0$
$\quad y \leftarrow 1 - x$
$\textbf{\textit{end}}$



System NFA



System NFA


i
[1, 0, 0]
1, 0, 0
[5, 0, 0]
5, 0, 0
[1, 1, 0]
1, 1, 0
[2, 1, 0]
2, 1, 0
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Property NFA

- Is there a full execution such that
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$$[\ell, x, 1][\ell, x, 1]^*[\ell, x, 0]^*[5, x, 0]$$

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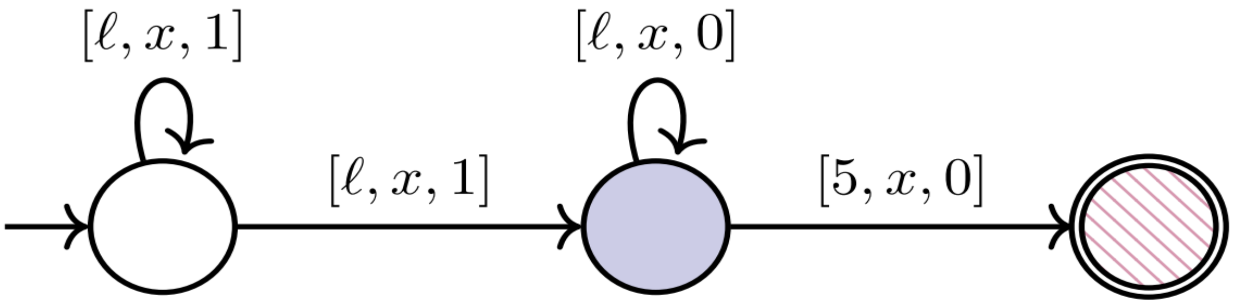
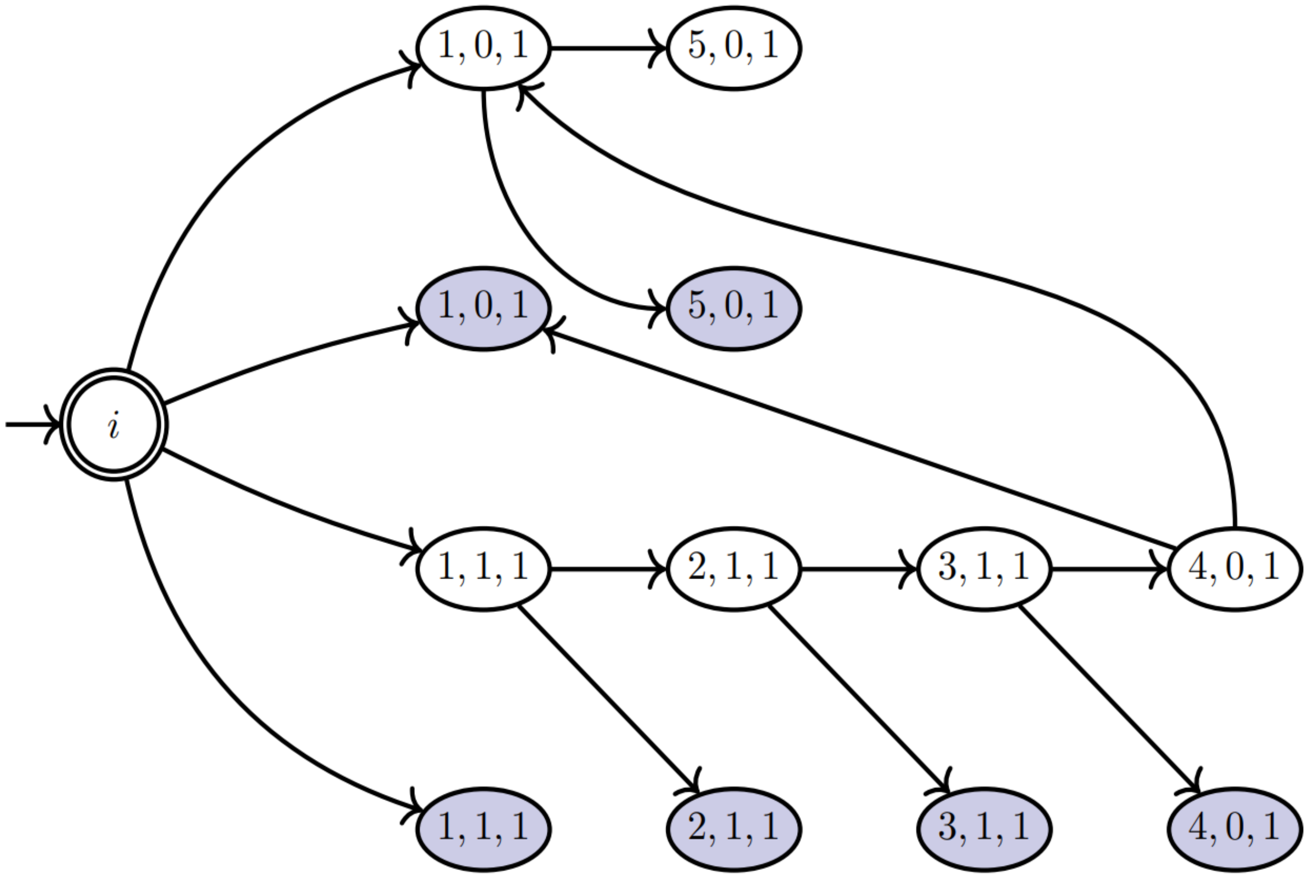
- Automaton for this set:



Intersection of the system and property NFAs



Automaton is empty, and so no execution satisfies the property

Another property

- Is the assignment $y \leftarrow x - 1$ redundant?

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- Potential executions where the assignment is executed at least once and it changes the value of y:

$$[\ell, x, y]^* \big([4, x, 0][1, x, 1] + [4, x, 1][1, x, 0]\big) [\ell, x, y]^*$$

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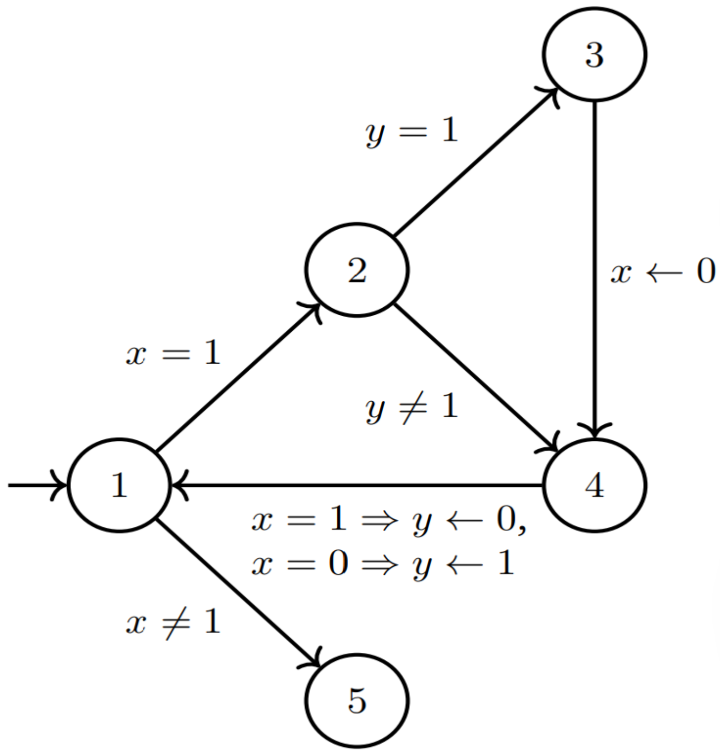
- Potential executions where the assignment is executed at least once and it changes the value of y:

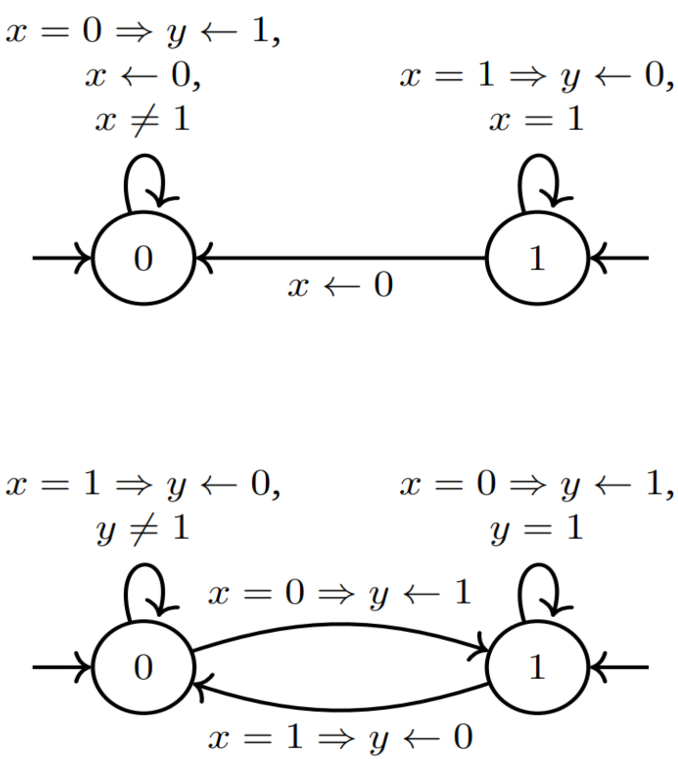
$$[\ell, x, y]^* \big([4, x, 0][1, x, 1] + [4, x, 1][1, x, 0]\big) [\ell, x, y]^*$$

- Therefore: assignment redundant iff none of these potential executions is a real execution of the program.

Networks of Automata

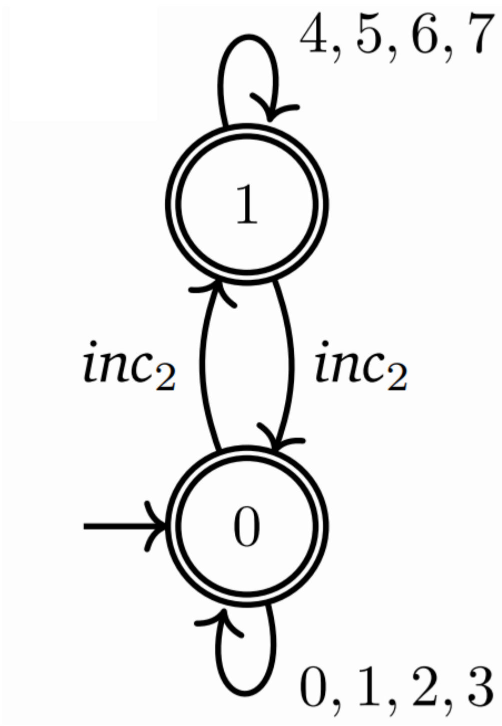
while $x = 1$ **do**
  if $y = 1$ **then**
    $x \leftarrow 0$
  $y \leftarrow 1 - x$
end



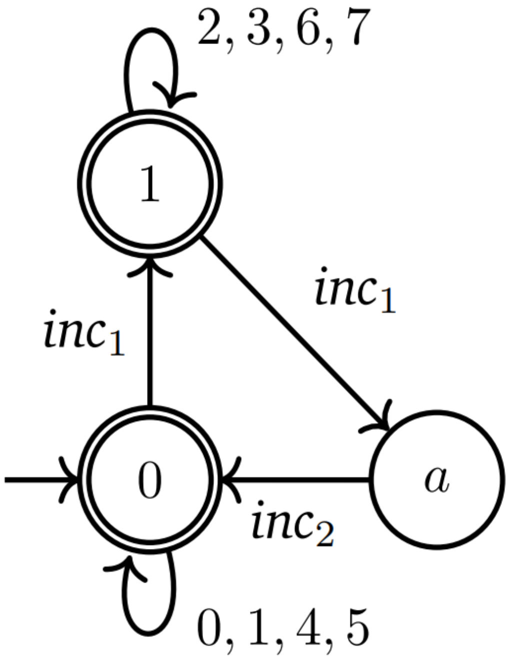


Networks of Automata

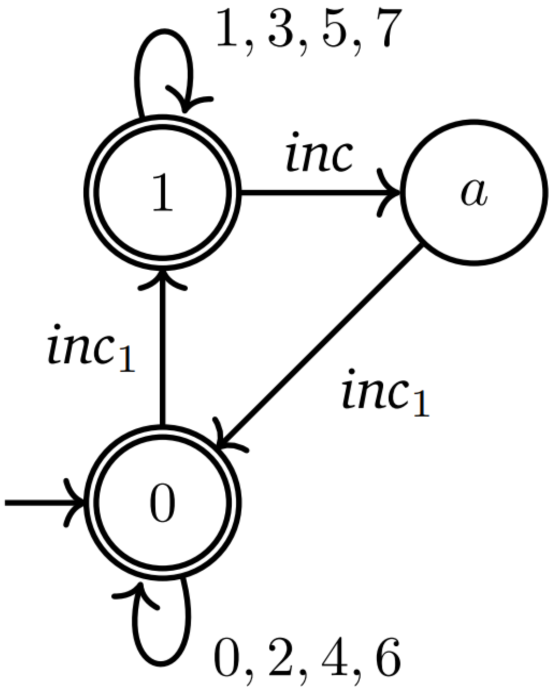
A_2



A_1



A_0

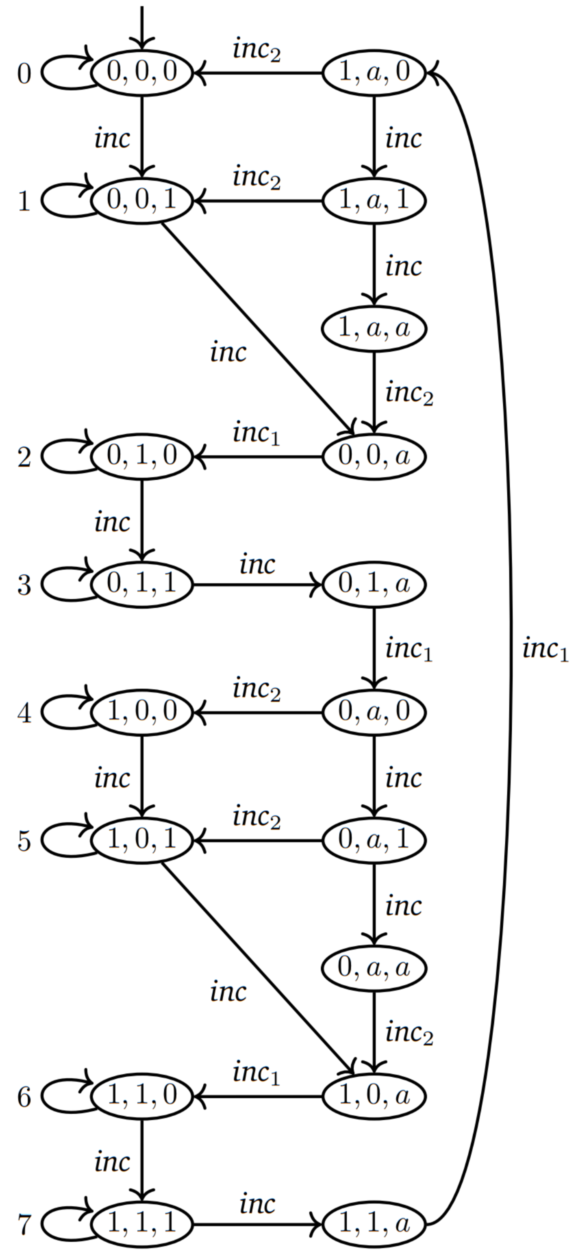


Networks of Automata

- Tuple $\mathcal{A} = \langle A_1, \ldots, A_n \rangle$ of NFAs.
- Each NFA has its own alphabet Σ_i of actions.
- Alphabets usually not disjoint!
- A_i participates in action a if $a \in \Sigma_i$.
- A configuration is a tuple $\langle q_1, \ldots, q_n \rangle$ of states, one for each automaton of the network.
- $\langle q_1, \ldots, q_n \rangle$ enables a if every participant in a is in a state from which an a-transition is possible.
- Enabled actions can occur, and their occurrence simultaneously changes the states of their participants. Non-participants stay idle and don't change their states.

Networks of Automata

Configuration graph of the network



Asynchronous product

AsyncProduct$(A_1, \ldots, A_n)$

Input: a network of automata $\mathcal{A} = \langle A_1, \ldots, A_n \rangle$, where
$A_i = (Q_i, \Sigma_i, \delta_i, Q_{0i}, F_i)$ for every $i \in \{1, \ldots, n\}$

Output: NFA $A_1 \otimes \cdots \otimes A_n = (Q, \Sigma, \delta, Q_0, F)$ recognizing $\mathcal{L}(\mathcal{A})$

$Q, \delta, F \leftarrow \emptyset$
$Q_0 \leftarrow Q_{01} \times \cdots \times Q_{0n}$
$W \leftarrow Q_0$
while $W \neq \emptyset$ **do**
    pick $[q_1, \ldots, q_n]$ **from** W
    add $[q_1, \ldots, q_n]$ **to** Q
    if $\bigwedge_{i=1}^{n} q_i \in F_i$ **then add** $[q_1, \ldots, q_n]$ **to** F
    for all $a \in \Sigma_1 \cup \ldots \cup \Sigma_n$ **do**
        for all $i \in [1..n]$ **do**
            if $a \in \Sigma_i$ **then** $Q'_i \leftarrow \delta_i(q_i, a)$ **else** $Q'_i = \{q_i\}$
        for all $[q'_1, \ldots, q'_n] \in Q'_1 \times \ldots \times Q'_n$ **do**
            if $[q'_1, \ldots, q'_n] \notin Q$ **then add** $[q'_1, \ldots, q'_n]$ **to** W
            add $([q_1, \ldots, q_n], a, [q'_1, \ldots, q'_n])$ **to** δ
return $Q, \Sigma_1 \cup \cdots \cup \Sigma_n, \delta, Q_0, F$

Networks of Automata

Concurrent programs as networks of automata: Lamport's 1-bit algorithm

Two processes $i = 0, 1$ with shared variables $b_0, b_1 \in \{0, 1\}$

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Process 0

  repeat

nc_0: $b_0 \leftarrow 1$

t_0: **while** $b_1 = 1$ **do skip**

c_0: $b_0 \leftarrow 0$

  forever

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Process 1
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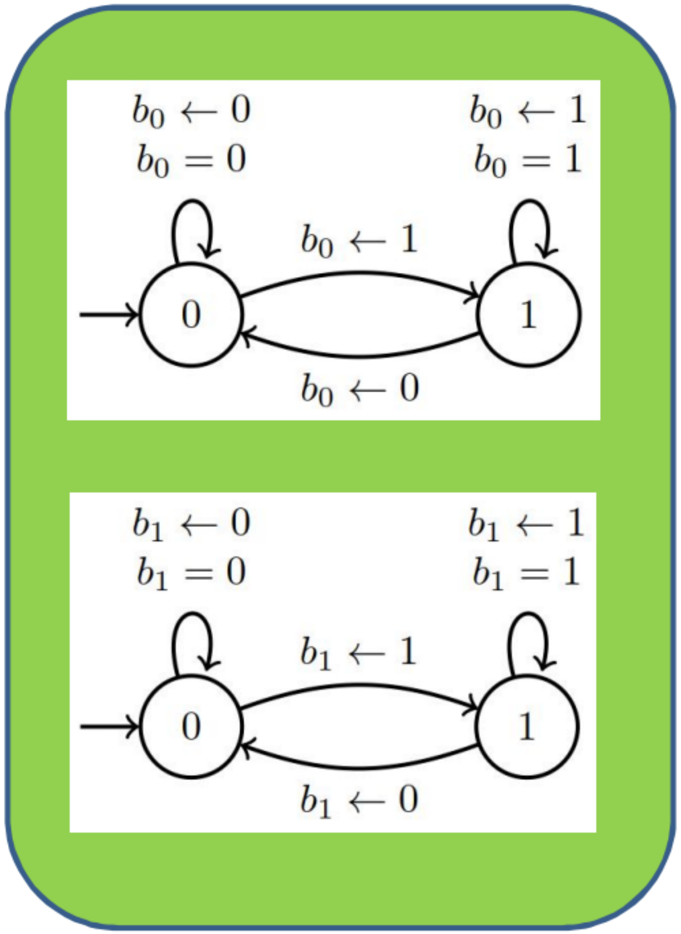
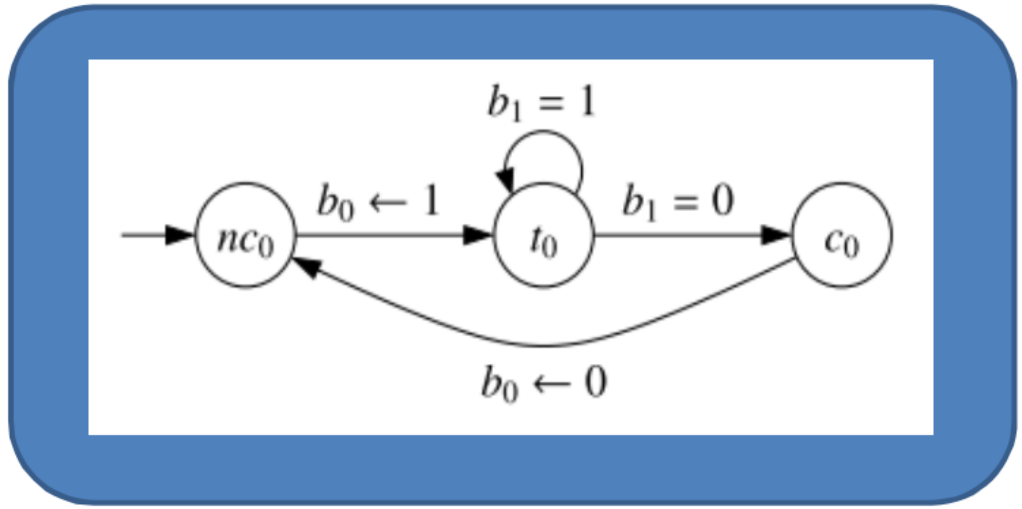
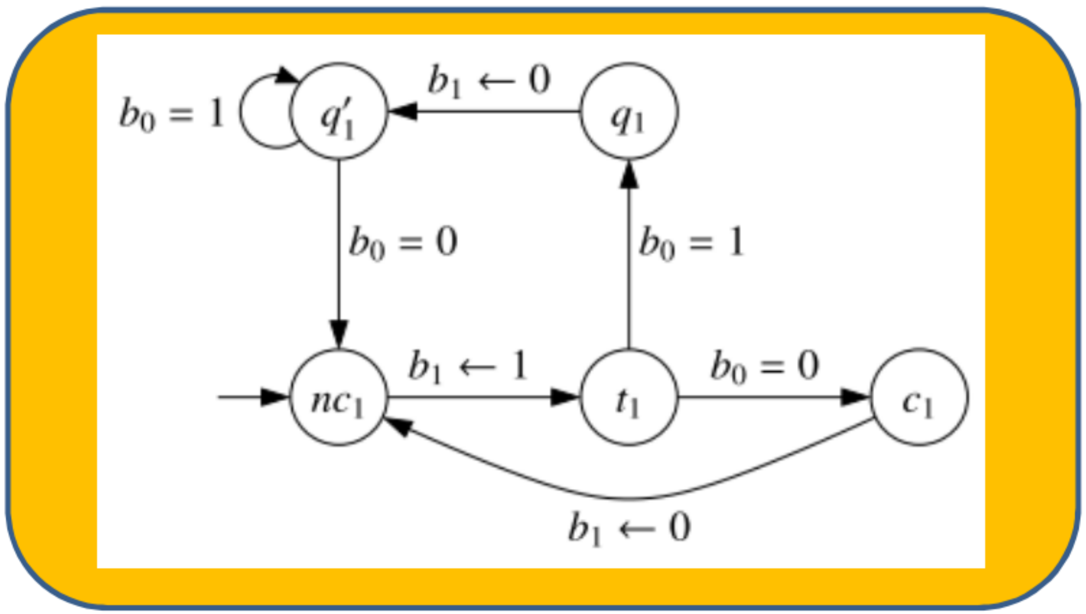
nc_1: $b_1 \leftarrow 1$
t_1: **if** $b_0 = 1$ **then**
q_1: $b_1 \leftarrow 0$
q_1': **while** $b_0 = 1$ **do skip**
  goto nc_1
c_1: $b_1 \leftarrow 0$ **forever**

Network for the two-process case







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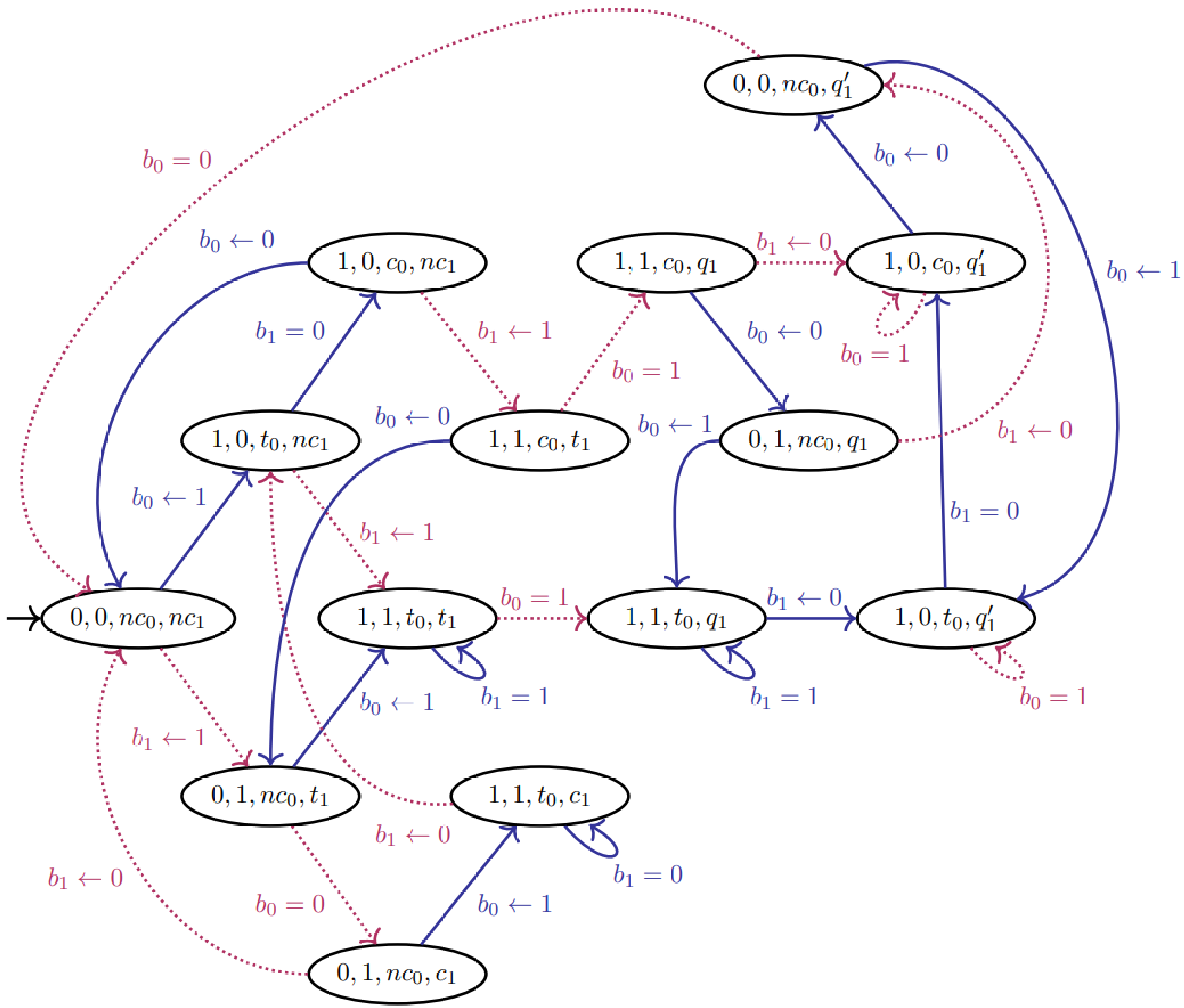

Checking properties of the algorithm

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- Deadlock freedom: every configuration has at least one successor.
- Mutual exclusion: no configuration of the form $[b_0, b_1, c_0, c_1]$ is reachable
- Bounded overtaking (for process 0): after process 0 signals interest in accessing the critical section, process 1 can enter the critical section at most once before process 0 enters.
  - Let NC_i, T_i, C_i be the configurations in which process i is non-critical, trying, or critical
  - Set of potential executions violating the property:

$$\Sigma^* T_0 (\Sigma \setminus C_0)^* C_1 (\Sigma \setminus C_0)^* NC_1 (\Sigma \setminus C_0)^* C_1 \Sigma^*$$

The issue of state explosion

In sequential programs, the number of reachable configurations grows exponentially in the number of variables.

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Proposition

The following problem is PSPACE-complete.

Given: a boolean program π (program with only boolean variables), and a NFA A_V recognizing a set of potential executions

Decide: Is $E_\pi \cap A_V$ empty?

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In concurrent programs, the number of reachable configurations also grows exponentially in the number of components.

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Proposition

The following problem is PSPACE-complete.

Given: a network of automata $\mathcal{A} = \langle A_1, \ldots, A_n \rangle$ and a NFA A_V recognizing a set of potential executions of $\mathcal{A}$

Decide: Is $L(A_1 \otimes \cdots \otimes A_n \otimes A_V) = \emptyset$?

On-the-fly verification

CheckViol$(A_1, \ldots, A_n, r_V)$

Input: a network $\mathcal{A} = \langle A_1, \ldots A_n \rangle$, where $A_i = (Q_i, \Sigma_i, \delta_i, Q_{0i}, F_i)$
a regular expression r_V over the configurations of $\mathcal{A}$

Output: **true** if $\mathcal{L}(A_1 \otimes \cdots \otimes A_n) \cap \mathcal{L}(r_V)$ is nonempty, **false** otherwise

$(Q_V, \Sigma_V, \delta_V, Q_{0V}, F_V) \leftarrow \mathit{REtoNFA}(r_V)$
$Q \leftarrow \emptyset$; $Q_0 \leftarrow Q_{01} \times \cdots \times Q_{0n} \times Q_{0V}$
$W \leftarrow Q_0$
while $W \neq \emptyset$ **do**
  pick $[q_1, \ldots, q_n, q]$ **from** W
  add $[q_1, \ldots, q_n, q]$ **to** Q
  for all $a \in \Sigma_1 \cup \ldots \cup \Sigma_n$ **do**
    for all $i \in [1..n]$ **do**
      if $a \in \Sigma_i$ **then** $Q'_i \leftarrow \delta_i(q_i, a)$ **else** $Q'_i = \{q_i\}$
    for all $[q'_1, \ldots, q'_n] \in Q'_1 \times \ldots \times Q'_n$ **do**
      $Q' \leftarrow \delta_V(q, [q'_1, \ldots, q'_n])$
      for all $q' \in Q'$ **do**
        if $\bigwedge_{i=1}^{n} q'_i \in F_i$ **and** $q' \in F_V$ **then return true**
        if $[q'_1, \ldots, q'_n, q'] \notin Q$ **then add** $[q'_1, \ldots, q'_n, q']$ **to** W
return false

Compositional verification

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- Replace A_1 by an automaton A'_1 recognizing $\mathrm{proj}_{\Sigma \setminus \Sigma_1}(L(A_1))$ and compute $A_{12} = A'_1 \otimes A_2$;
- Replace A_{12} by an automaton A'_{12} recognizing $\mathrm{proj}_{\Sigma \setminus (\Sigma_1 \cup \Sigma_2)}(L(A_{12}))$ and compute $A_{13} = A'_{12} \otimes A_3$;
- $\cdots$
- Replace $A_{1(n-1)}$ by an automaton $A'_{1(n-1)}$ recognizing $\mathrm{proj}_{\Sigma \setminus (\Sigma_1 \cup \cdots \cup \Sigma_{n-1})}(L(A_{1(n-1)}))$ and compute $A_{1n} = A'_{1(n-1)} \otimes A_n$.

Compositional verification

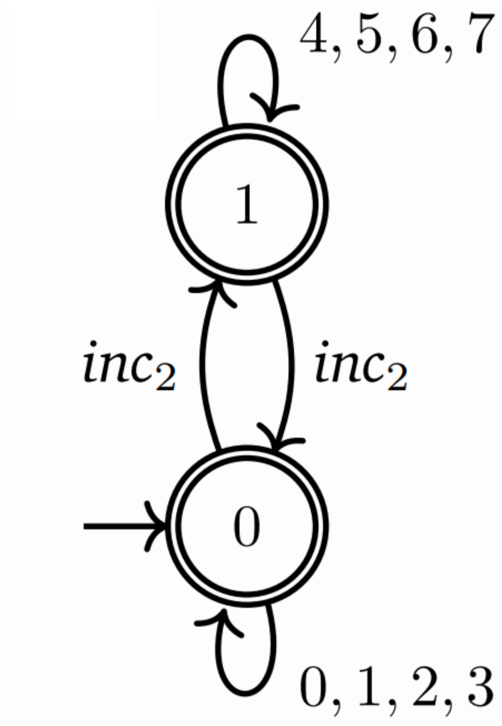
To check emptiness of an asynchronous product $A_1 \otimes \cdots \otimes A_n$ we can

- Replace A_1 by an automaton A_1' recognizing $\mathrm{proj}_{\Sigma \setminus \Sigma_1}(L(A_1))$ and compute $A_{12} = A_1' \otimes A_2$;
- Replace A_{12} by an automaton A_{12}' recognizing $\mathrm{proj}_{\Sigma \setminus (\Sigma_1 \cup \Sigma_2)}(L(A_{12}))$ and compute $A_{13} = A_{12}' \otimes A_3$;
- $\cdots$
- Replace $A_{1(n-1)}$ by an automaton $A_{1(n-1)}'$ recognizing $\mathrm{proj}_{\Sigma \setminus (\Sigma_1 \cup \cdots \cup \Sigma_{n-1})}(L(A_{1(n-1)}))$ and compute $A_{1n} = A_{1(n-1)}' \otimes A_n$.

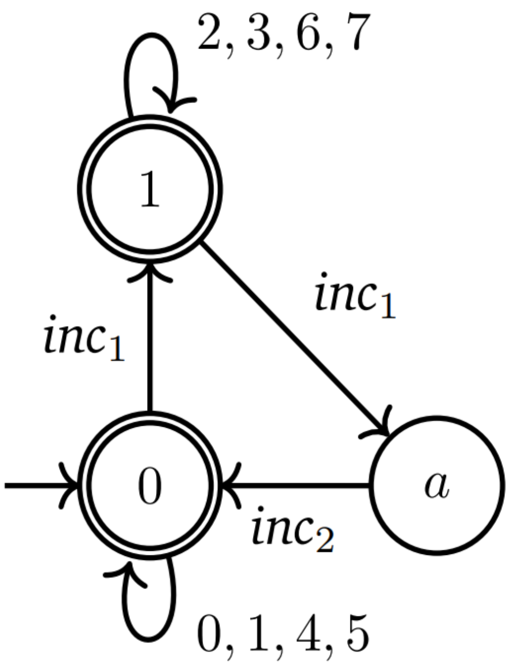
This can save space w.r.t. the direct computation.

Compositional verification

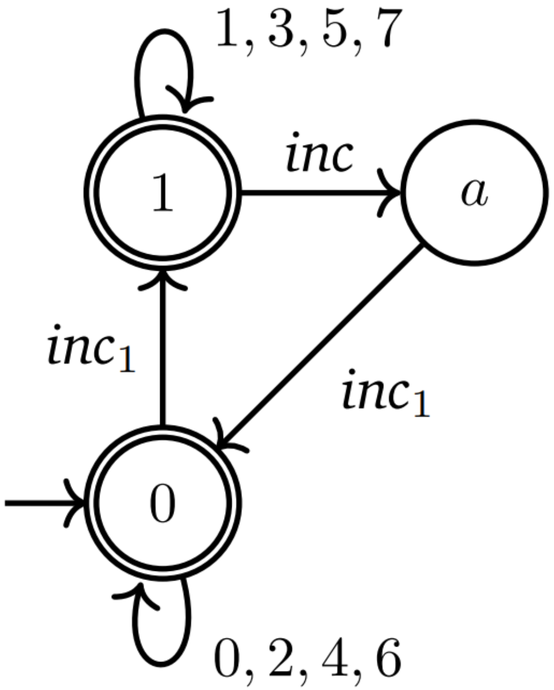
A_2



A_1

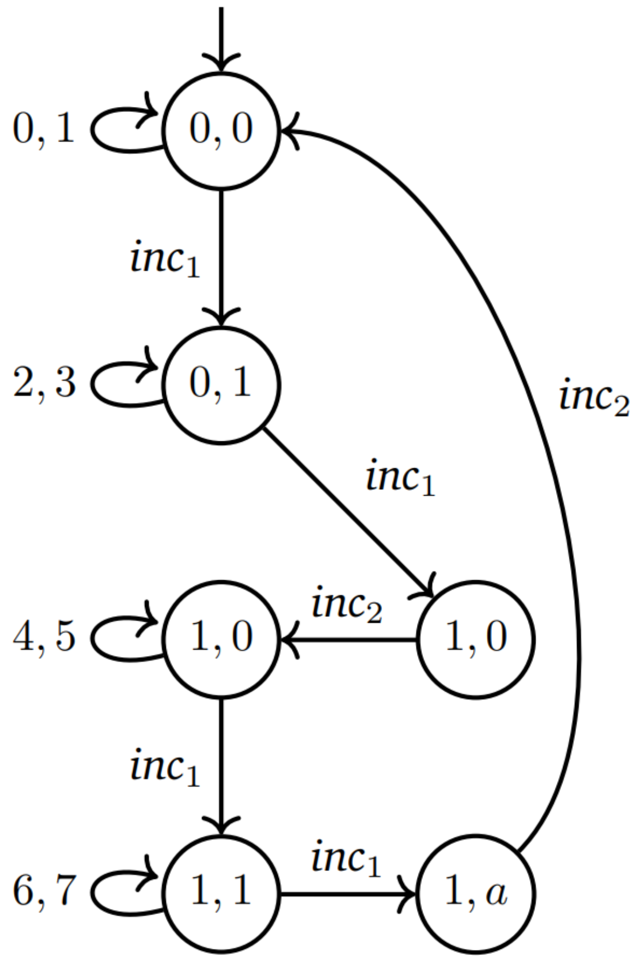


A_0

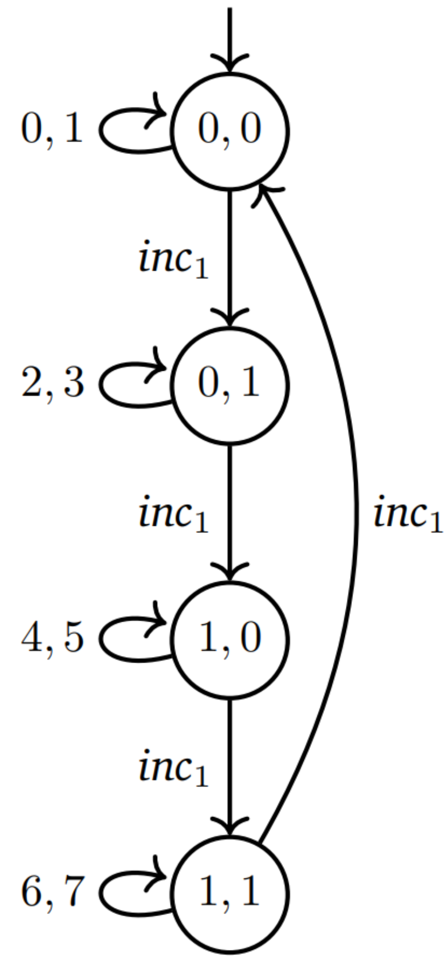


Compositional verification

A_{21}

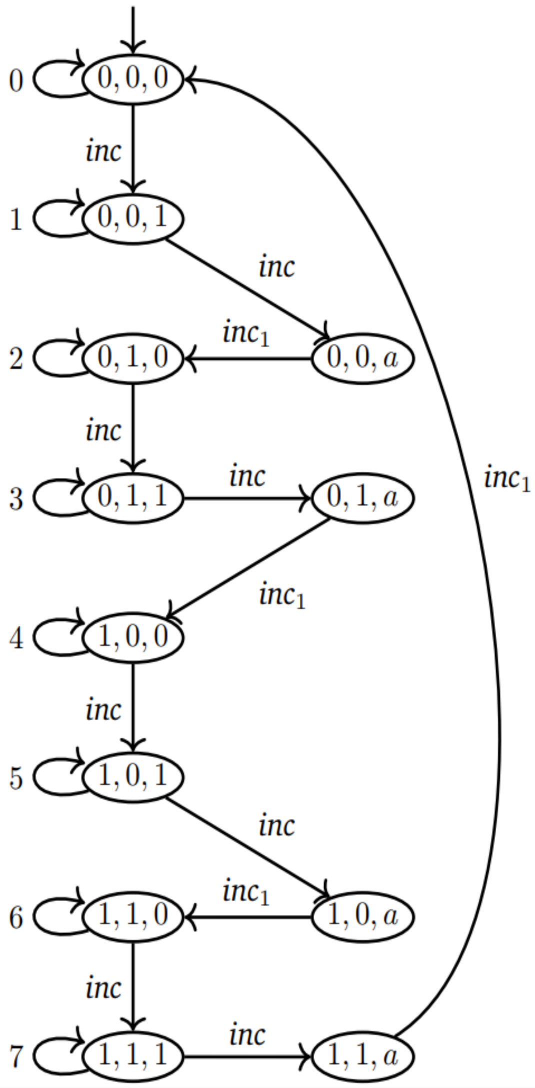


A'_{21}

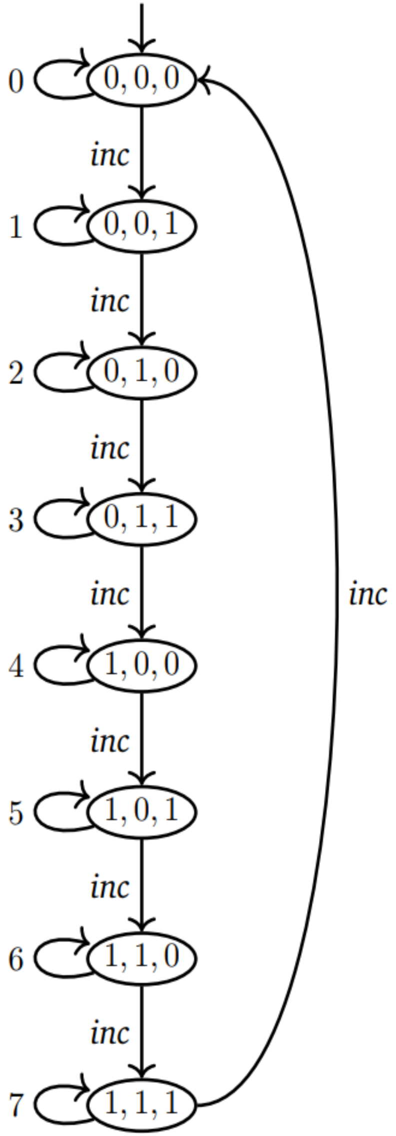


Compositional verification

A_{21}



A'_{21} (proj. on visible actions)

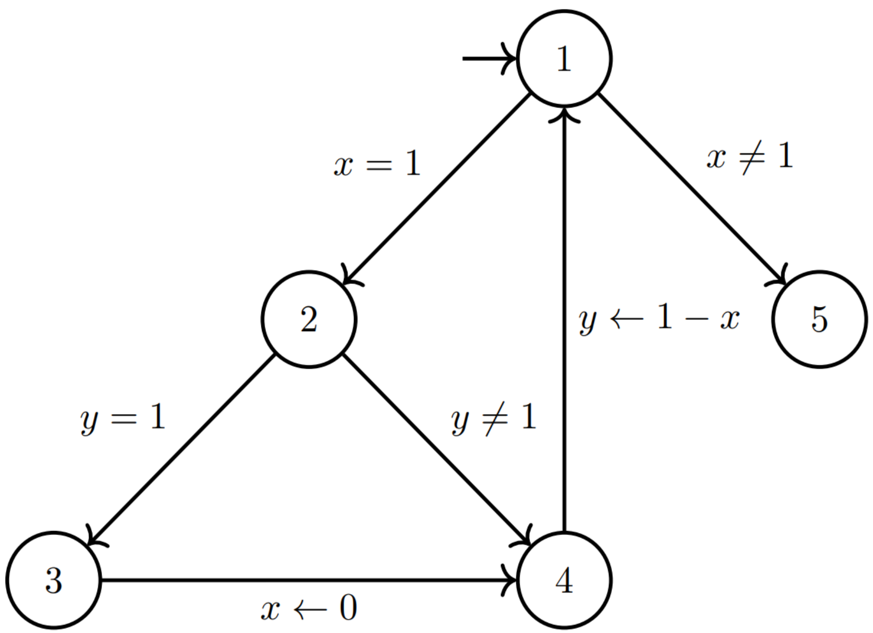


Symbolic exploration

- A technique to mitigate the issue of state explosion
- Configurations can be encoded as words.
- The set of reachable configurations of a program can be encoded as a language.
- We use automata to compactly store the set of reachable configurations.

Flowgraphs

```
while x = 1 do
   if y = 1 then
      x ← 0
   y ← 1 − x
end
```



Step relations

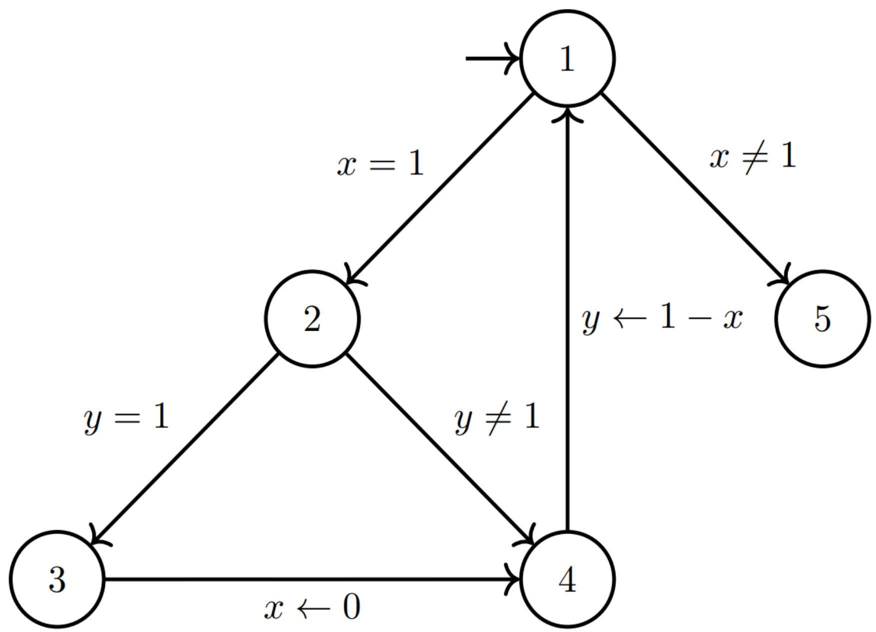
Let ℓ, ℓ' be two control points of a flowgraph.

The step relation $S_{\ell,\ell'}$ contains all pairs

$$([\ell, x_0, y_0], [\ell', x_0', y_0'])$$

of configurations such that: if at point ℓ the current values of x, y are x_0, y_0, then the program can take a step, after which the new control point is ℓ' and the new values of x, y are x_0', y_0'.

Step relations



$$S_{\ell,\ell'} = \{([4, x_0, y_0], [1, x_0, 1 - x_0]) \mid x_0, y_0 \in \{0, 1\}\}$$

The global step relation S is the union of the step relations $S_{\ell,\ell'}$ for all pairs ℓ, ℓ' of control points.

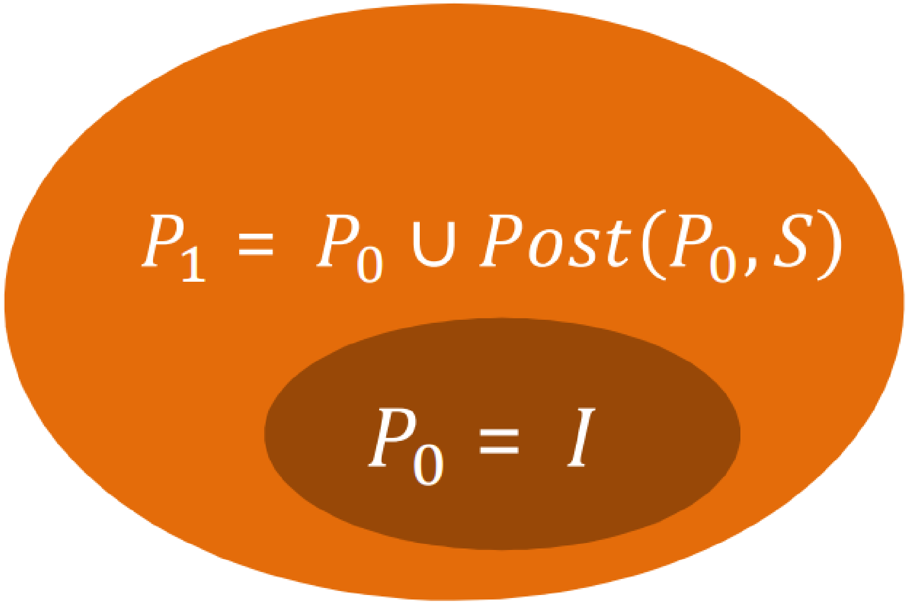
Computing reachable configurations

- Start with the set of initial configurations
- Iteratively: add the set of successors of the current set of configurations until a fixed point is reached

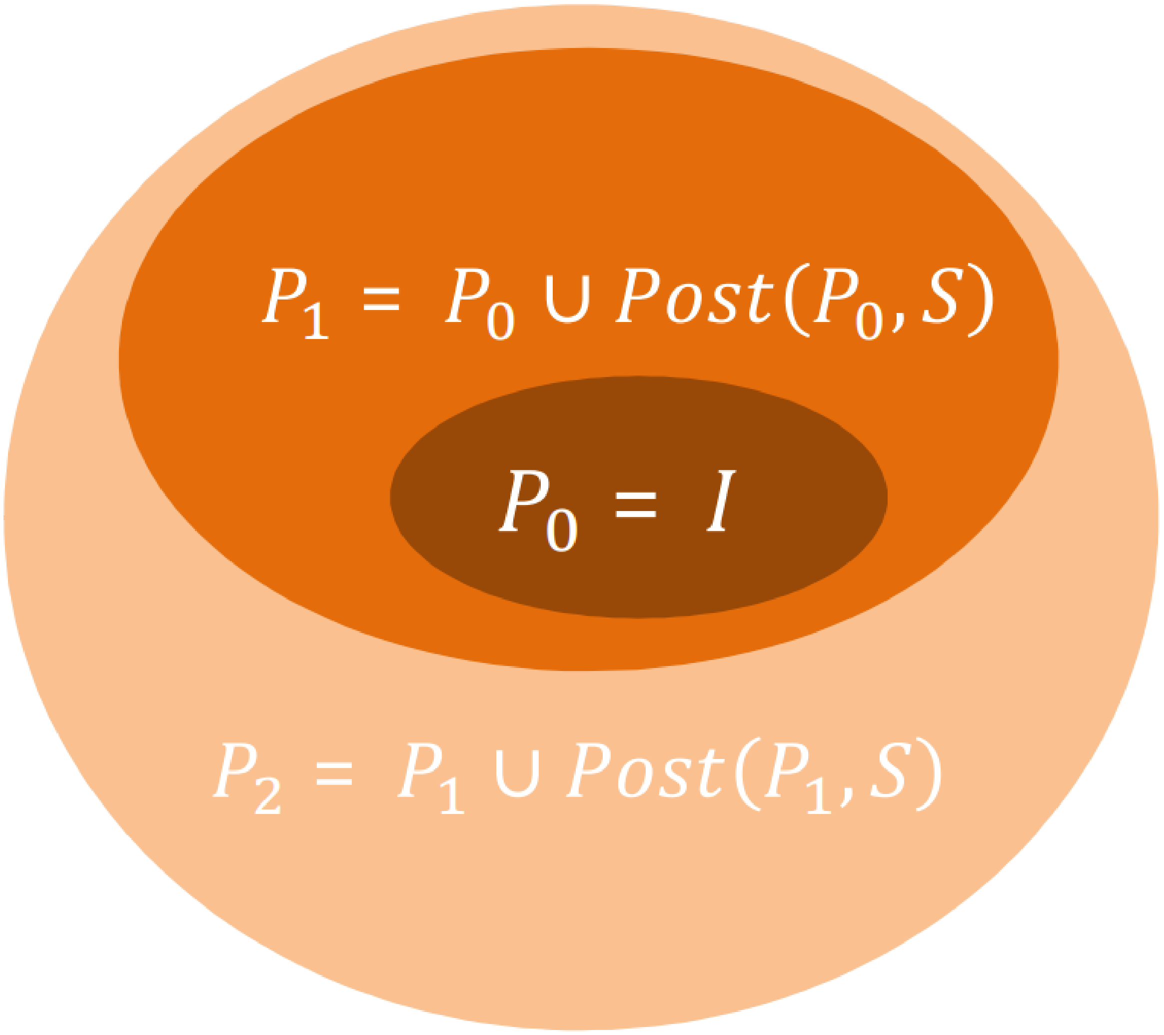
Computing reachable configurations

$P_0 = I$

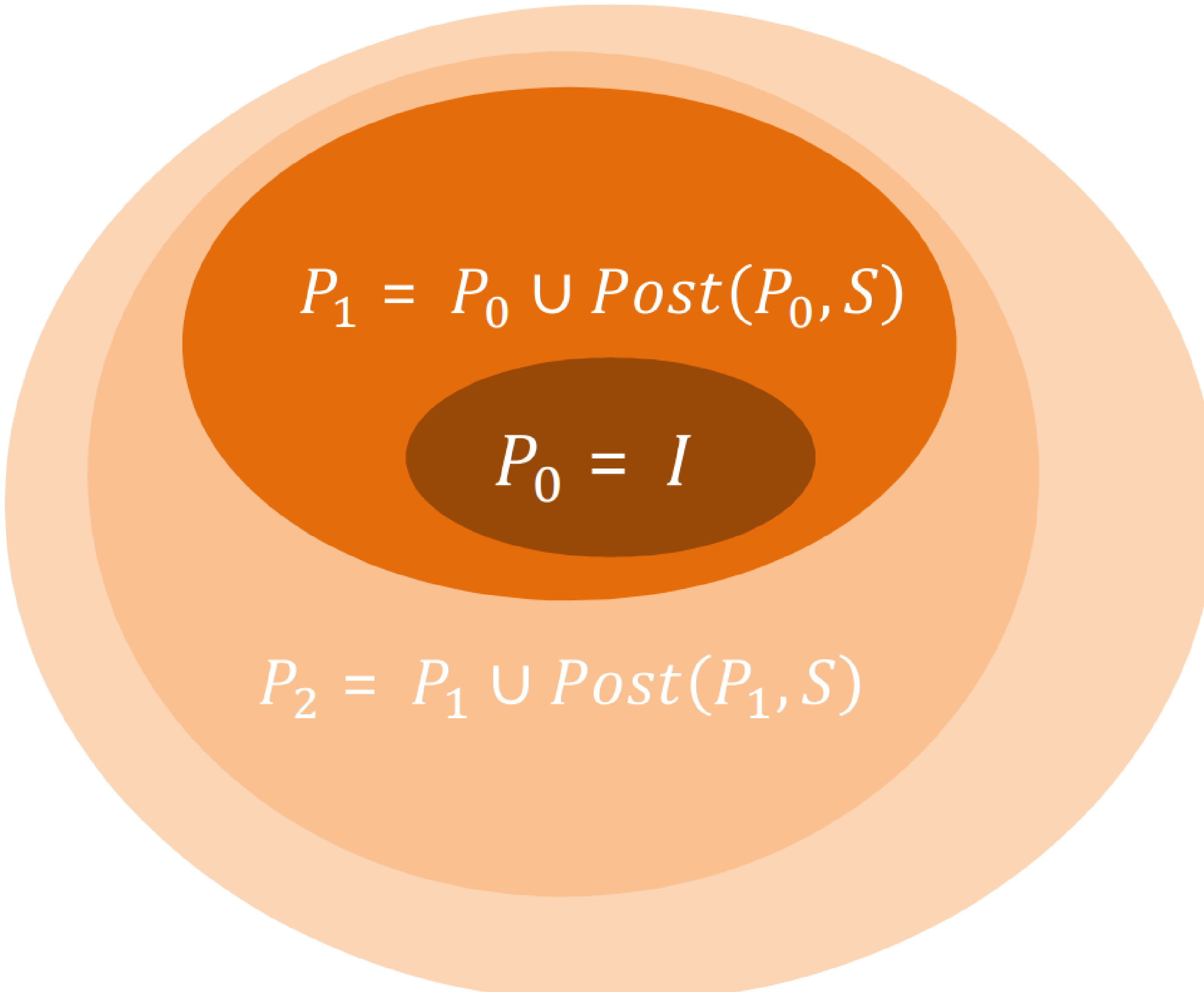
Computing reachable configurations



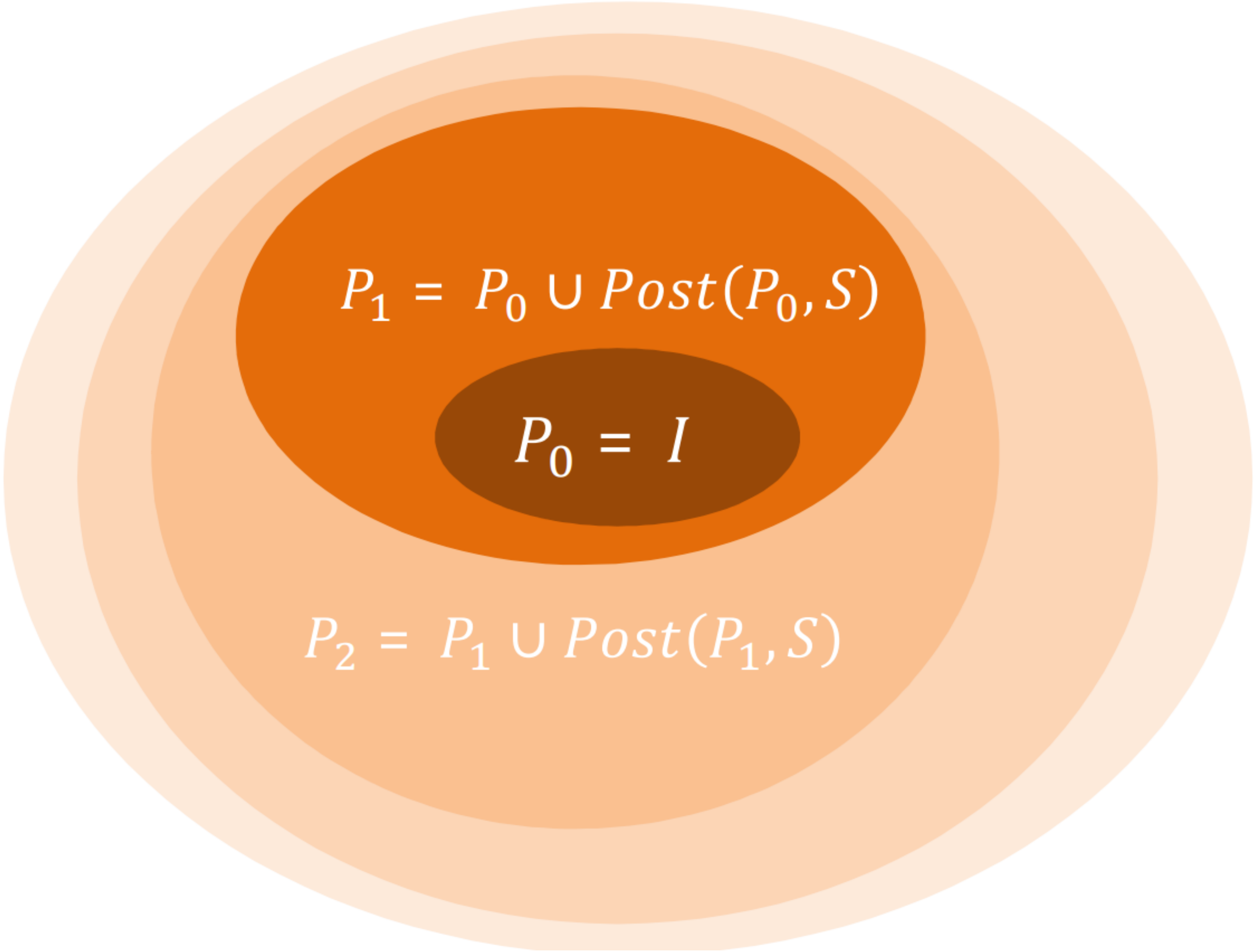
Computing reachable configurations



Computing reachable configurations



Computing reachable configurations

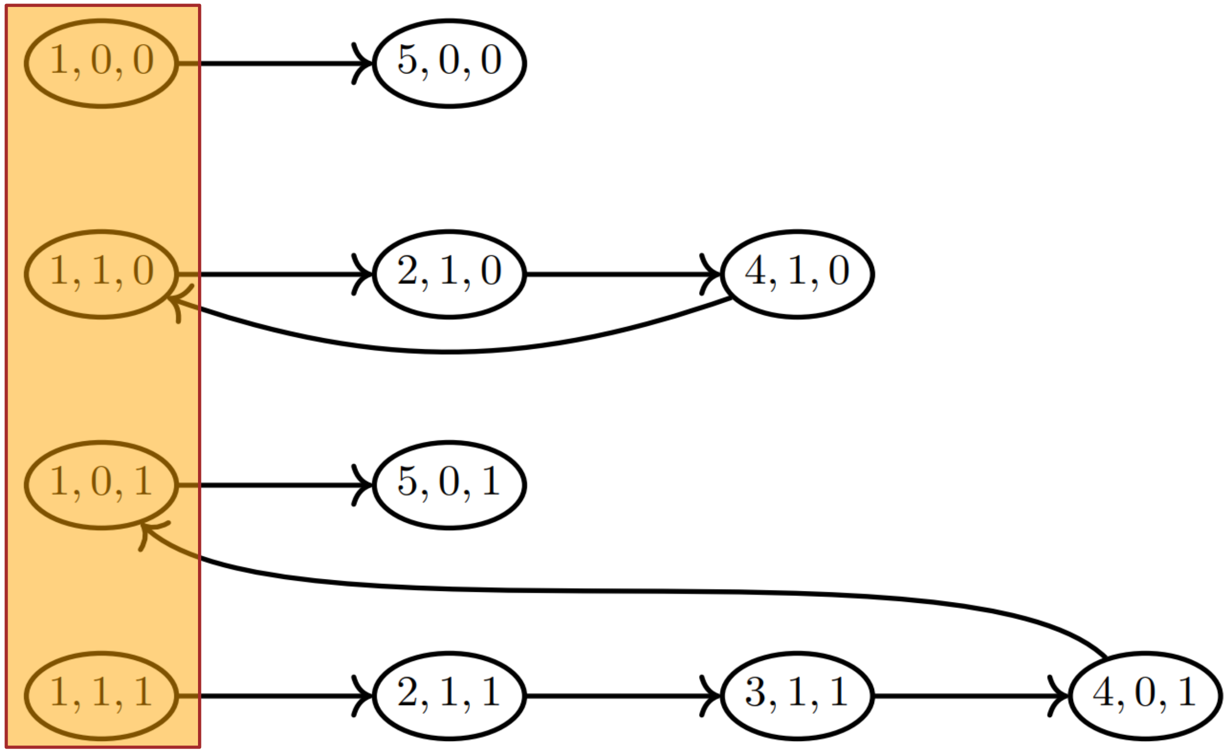


Computing reachable configurations

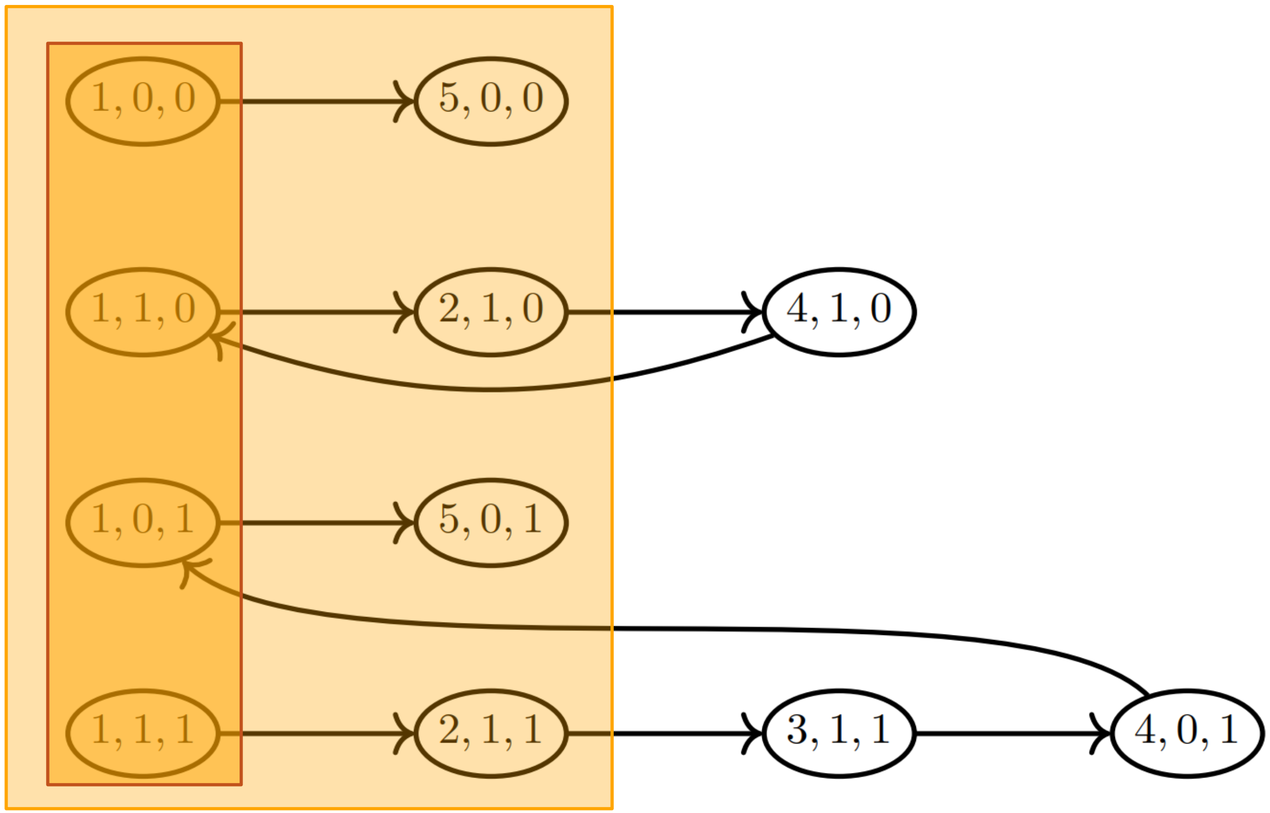
Reach(I, R)
Input: set I of initial configurations; step relation S
Output: set of configurations reachable from I

```
OldP ← ∅; P ← I
while P ≠ OldP do
    OldP ← P
    P ← Union(P, Post(P, S))
return P
```

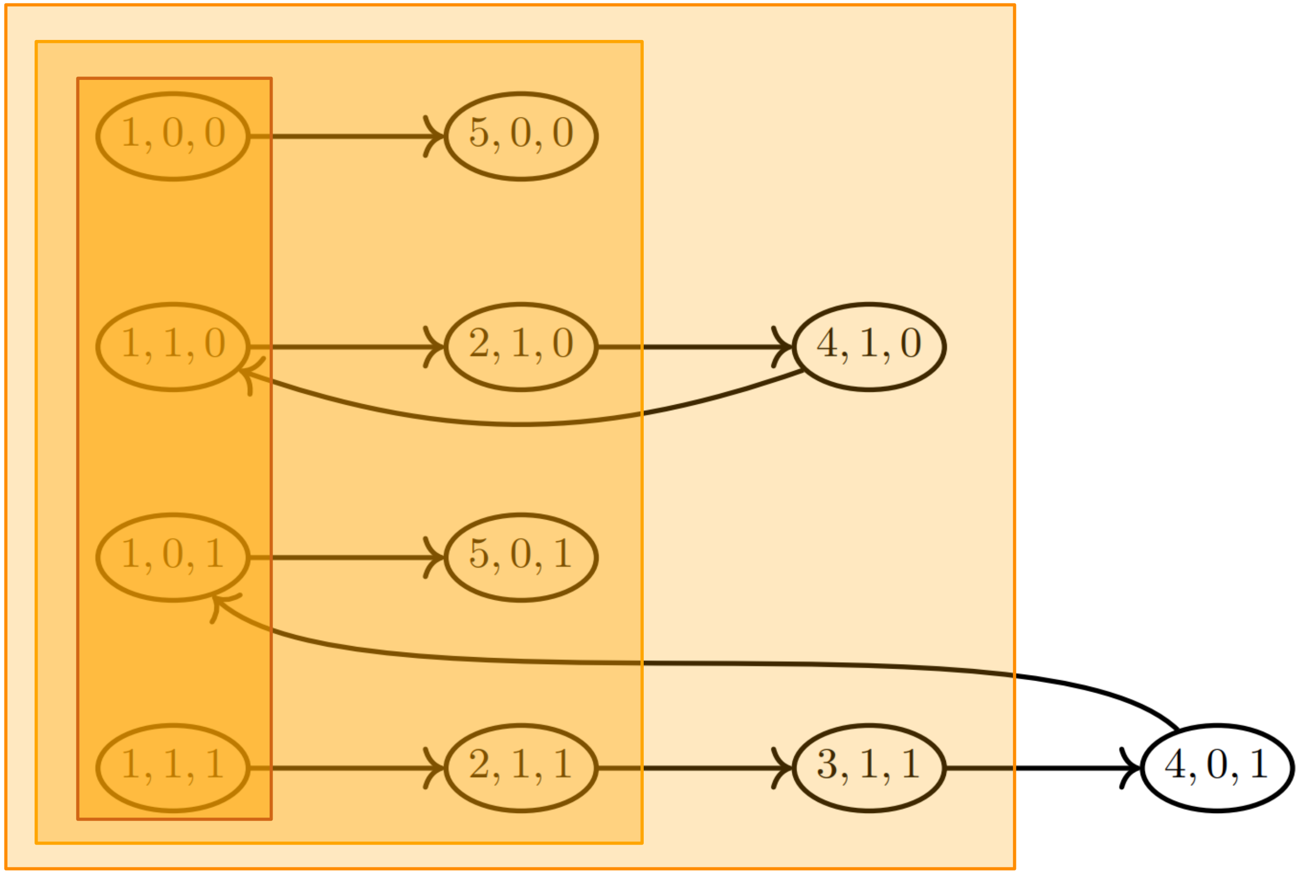
Computing reachable configurations



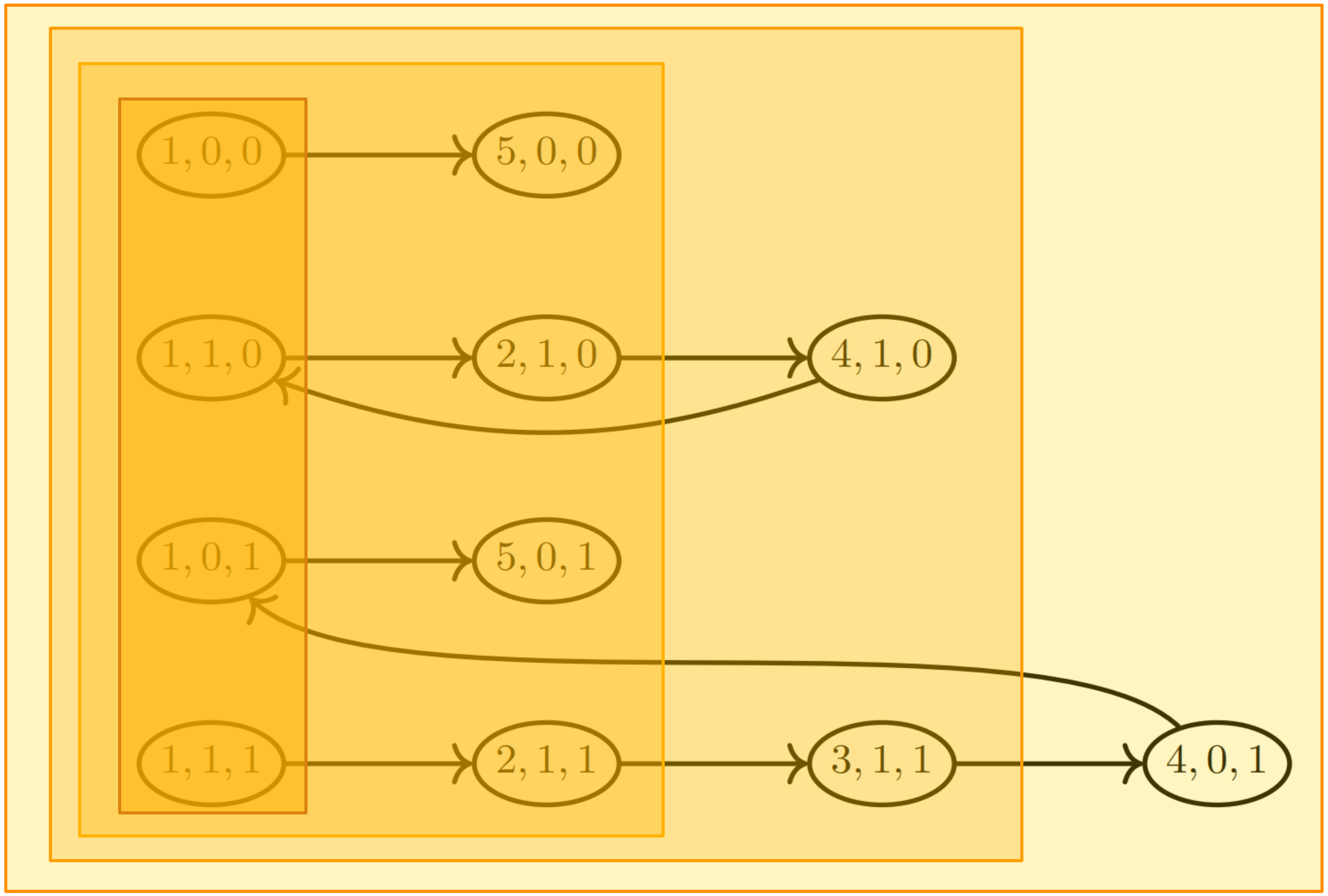
Computing reachable configurations



Computing reachable configurations

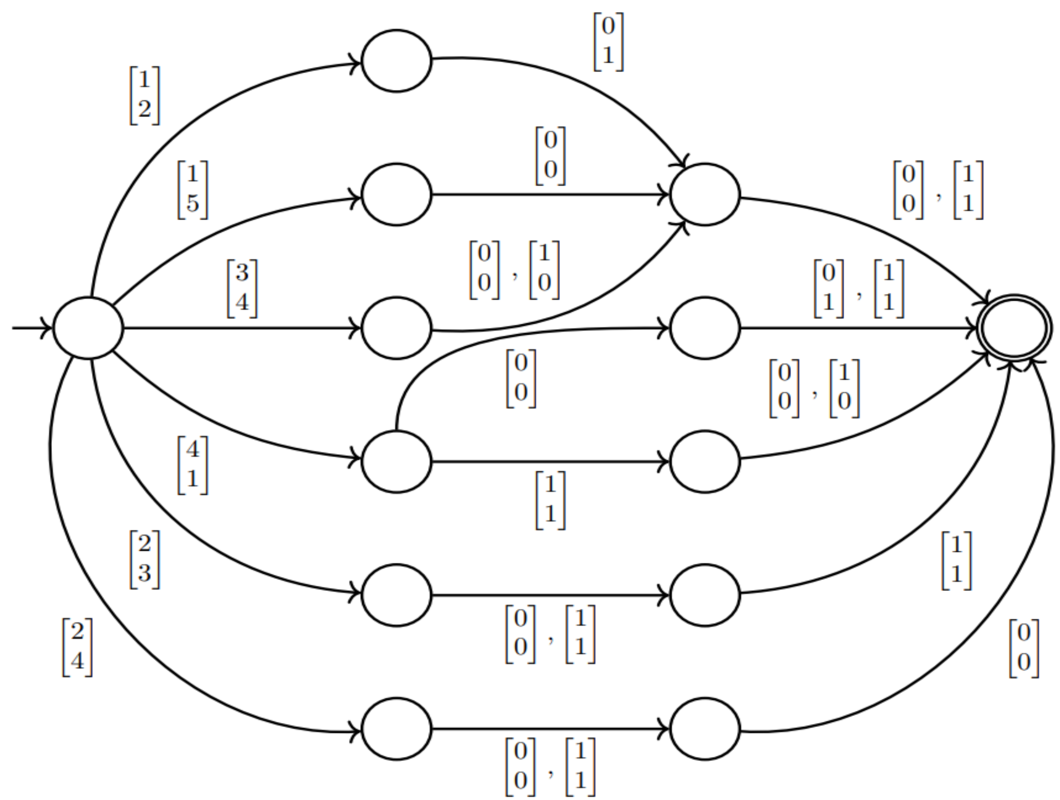


Computing reachable configurations

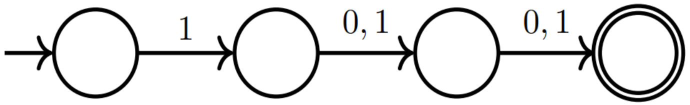


Example: Transducer for the global step relation

while $x = 1$ ***do***
  if $y = 1$ ***then***
    $x \leftarrow 0$
  $y \leftarrow 1 - x$
end

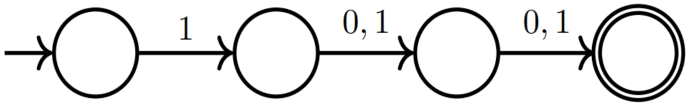


Example: DFAs generated by Reach

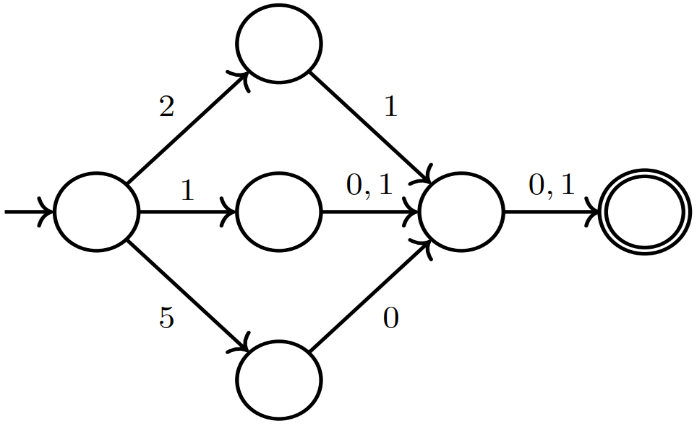


Initial configurations

Example: DFAs generated by Reach

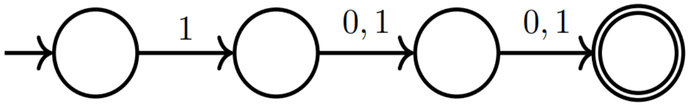


Initial configurations

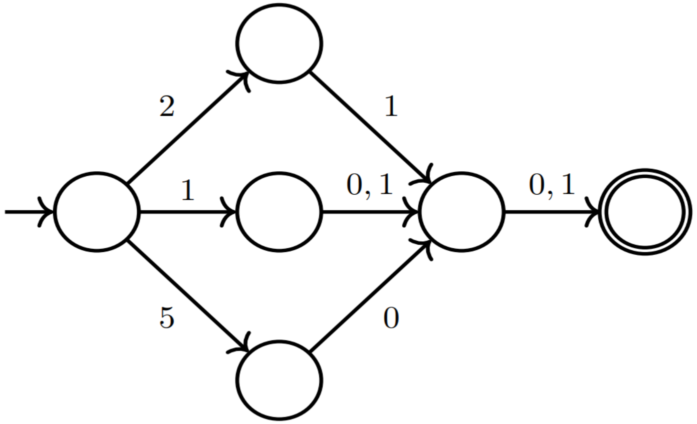


Configurations reachable in at most 1 step

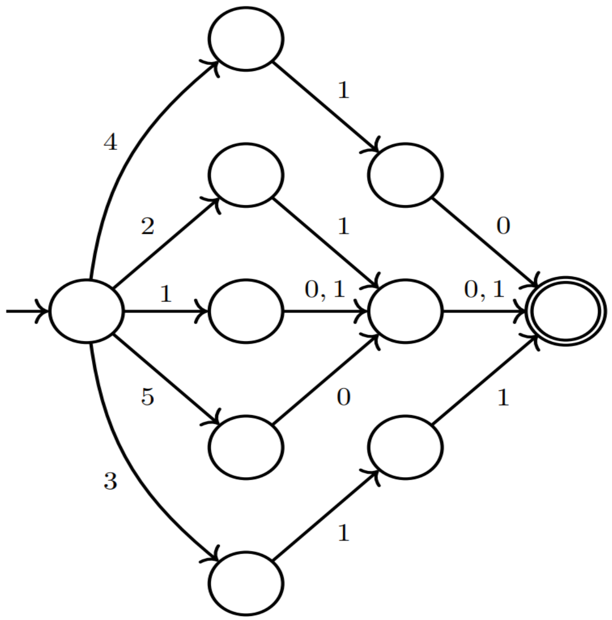
Example: DFAs generated by Reach



Initial configurations

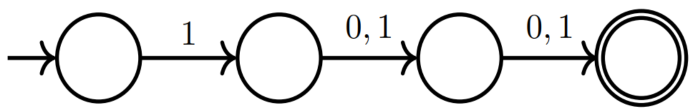


Configurations reachable in at most 1 step

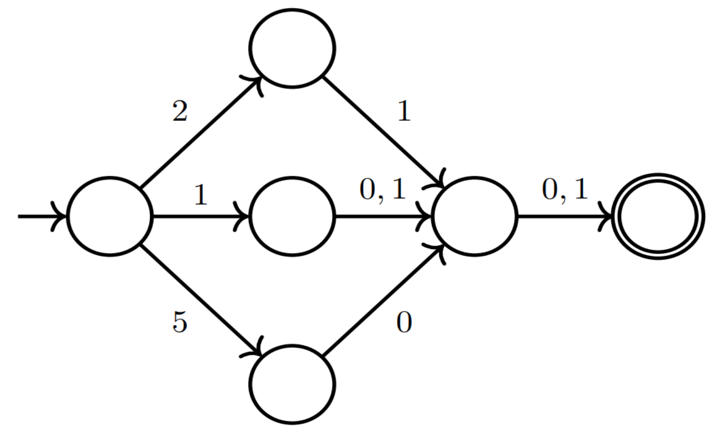


Configurations reachable in at most 2 steps

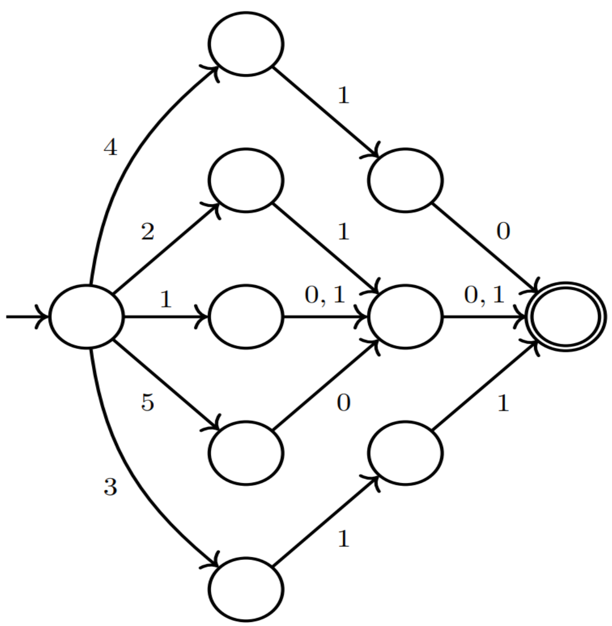
Example: DFAs generated by Reach



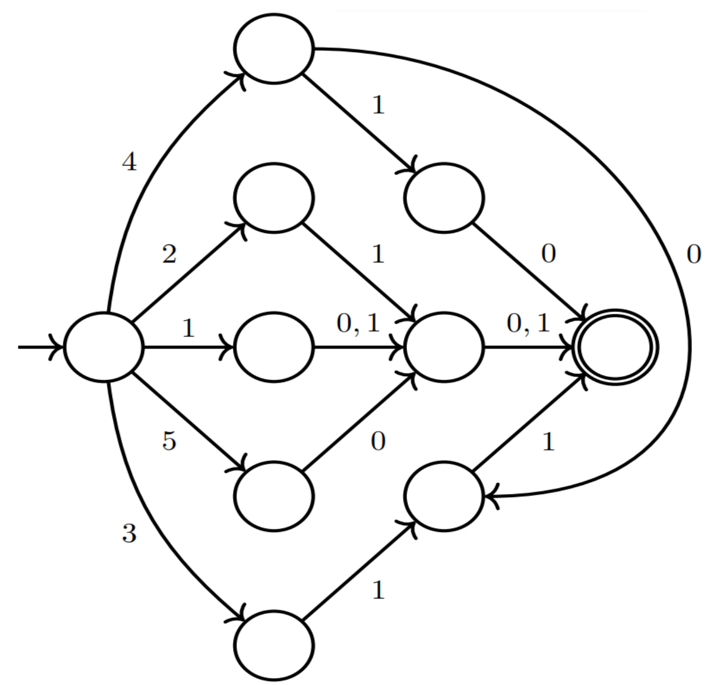
Initial configurations



Configurations reachable in at most 1 step



Configurations reachable in at most 2 steps



Configurations reachable in at most 3 steps

Variable orders

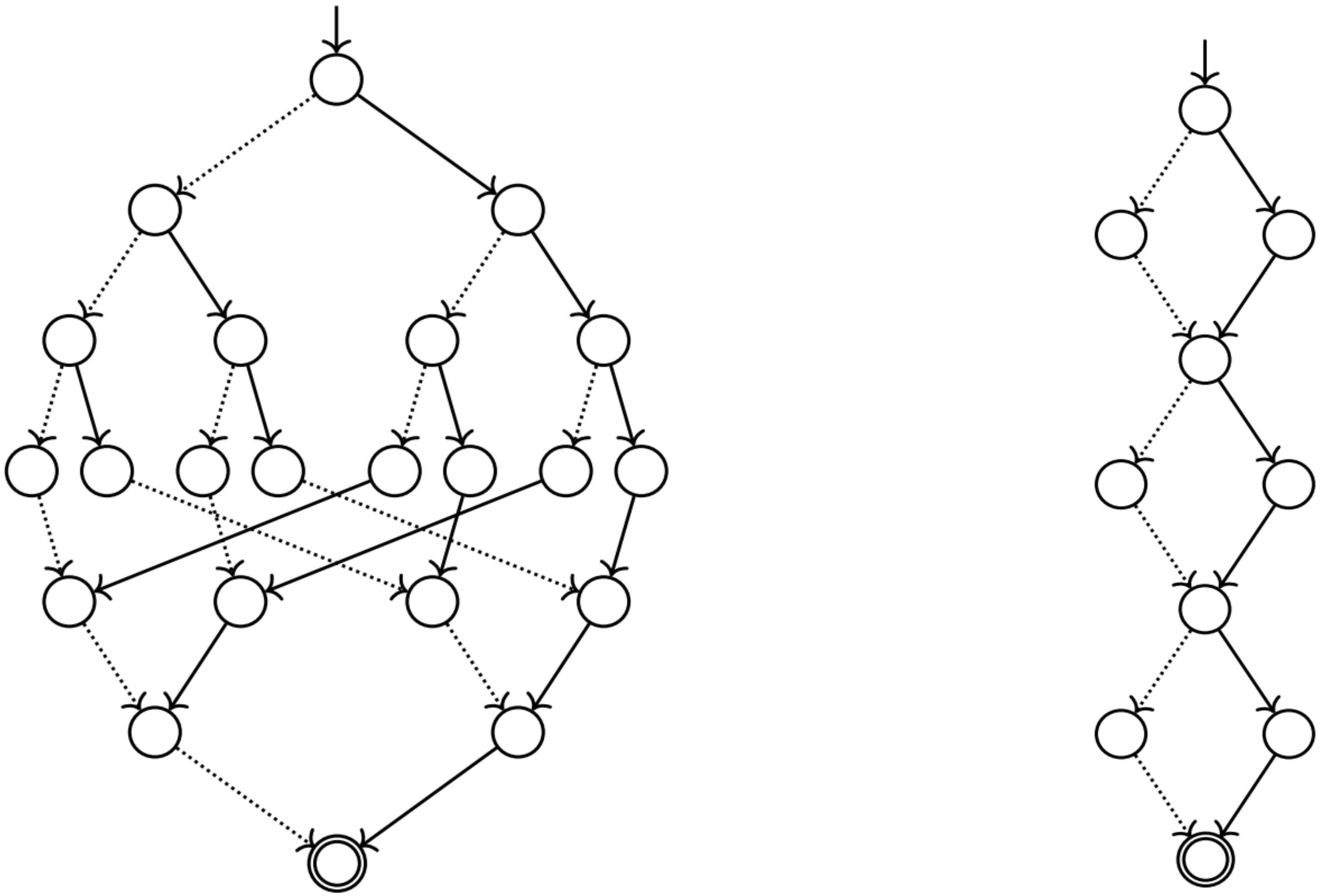
- Consider the set Y of tuples $[x_1, \ldots, x_{2k}]$ of booleans such that $x_1 = x_{k+1}, x_2 = x_{k+2}, \ldots, x_k = x_{2k}$

- A tuple $[x_1, \ldots, x_{2k}]$ can be encoded by the word $x_1 x_2 \cdots x_{2k-1} x_{2k}$ but also by the word $x_1 x_{k+1} \cdots x_k x_{2k}$

- For $k = 3$, the encodings of Y are then, respectively $\{000000, 001001, 010010, 011011, 100100, 101101, 110110, 111111\}$, $\{000000, 000011, 001100, 001111, 110000, 110011, 111100, 111111\}$.

- The minimal DFAs for these languages have very different sizes!

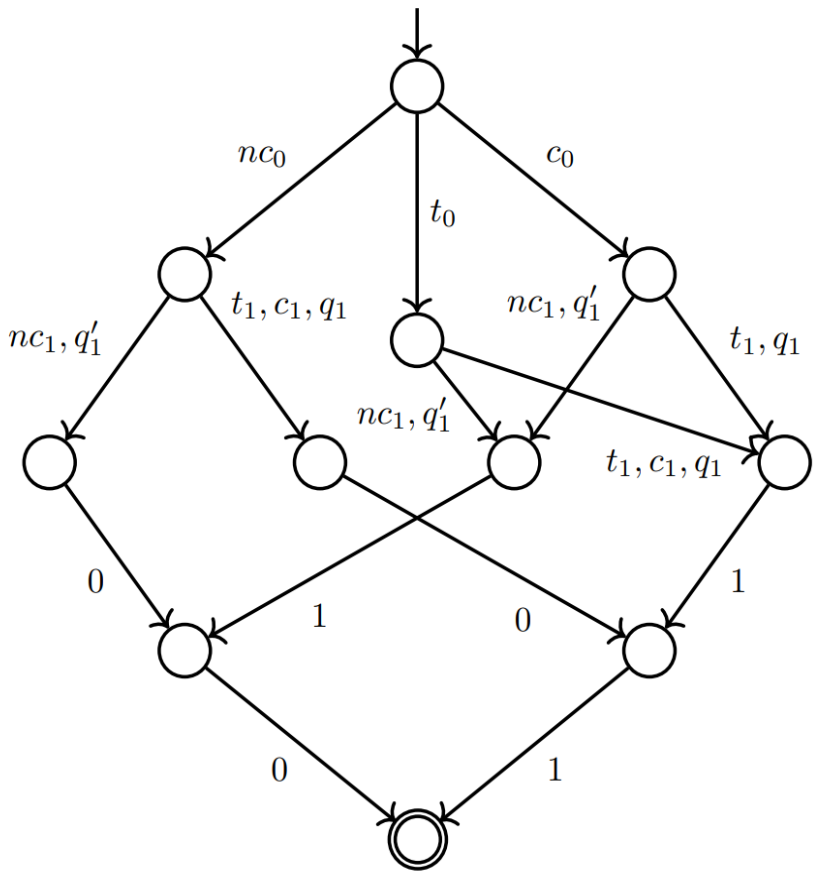
Minimal DFAs for Reachable Configurations



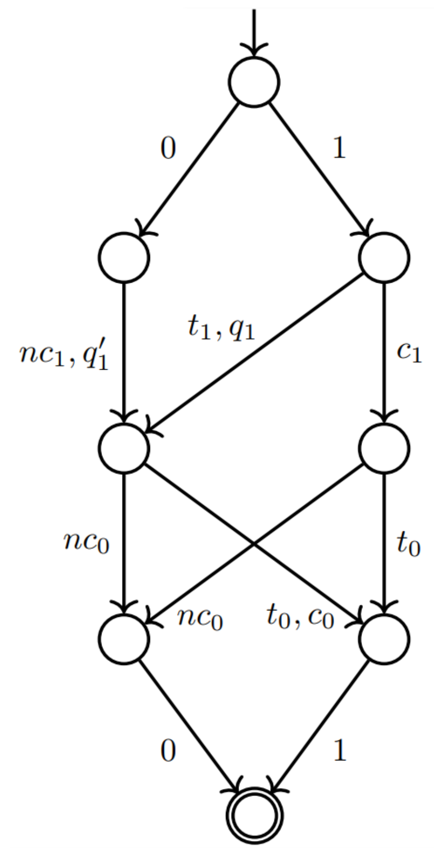
0 and 1 are respectively represented by solid and dotted arcs

Minimal DFAs for Reachable Configurations

Another example: Lamport's algorithm



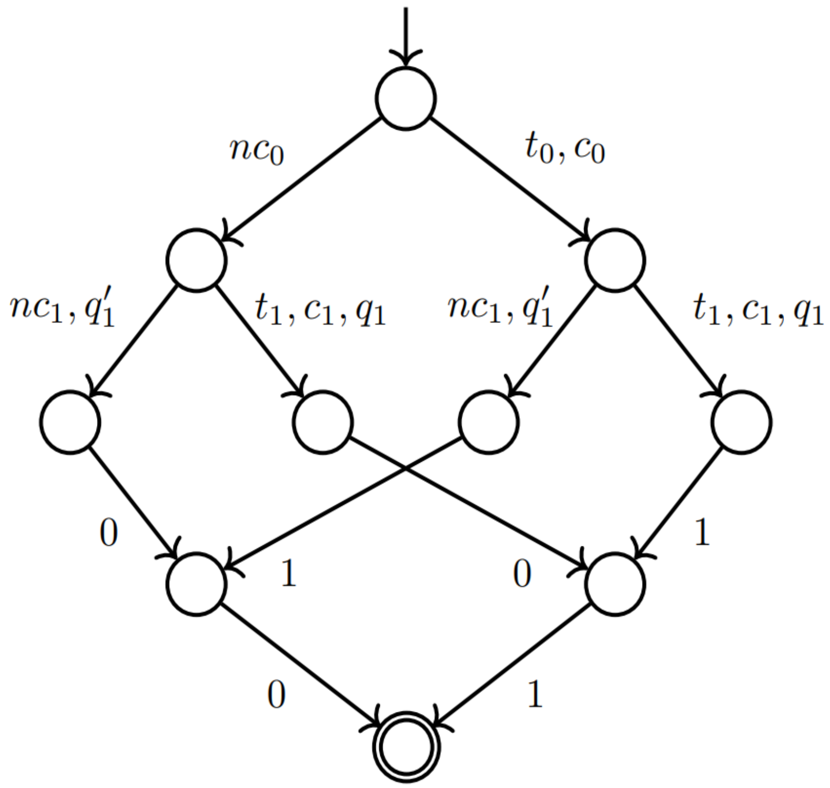
$\langle v_0, v_1, s_0, s_1 \rangle$ ensoded by $s_0 s_1 v_0 v_1$

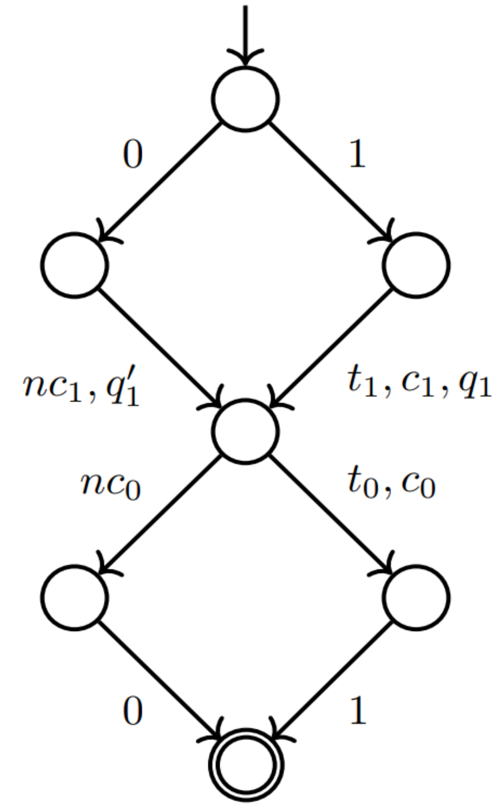


$\langle v_0, v_1, s_0, s_1 \rangle$ ensoded by $v_1 s_1 s_0 v_0$

Minimal DFAs for Reachable Configurations

Larger sets can yield smaller DFAs!





DFAs after adding the configuration $\langle c_0, c_1, 1, 1\rangle$ to the set

Reachable Configurations

- When encoding configurations, good variable orders can lead to much smaller automata.

- Unfortunately, the problem of finding an optimal encoding for a language represented by a DFA is NP-complete